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Abstract

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MATHEMATICS

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ON THE ERGODIC PROPERTY OF NONHOMOGENEOUS DIFFUSION PROCESSES

(Presented by Academician I. G. Petrovsky, 22 III 1962)

Let the operator

$$L(t, x) = \sum_{i,j=1}^N a_{ij}(t, x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^N b_i(t, x) \frac{\partial}{\partial x_i}$$

be defined everywhere in the domain $\{x \in E_N\} \times \{-\infty < t < \infty\}$ (E_N is N -dimensional Euclidean space). Suppose that the coefficients satisfy the conditions:

$$\sum_{i,j=1}^N a_{ij}(t, x) \xi_i \xi_j \geq \gamma(x) \sum_{i=1}^N \xi_i^2$$

for all real ξ_i and for some continuous positive function $\gamma(x)$;

$$|a_{ij}(t, x)| \leq M(r^2 + 1), \quad |b_i(t, x)| < M(r + 1)$$

for all t , where M is a constant, and

$$r^2 = \sum_{i=1}^N x_i^2 = |x|^2.$$

The coefficients are also assumed to be sufficiently smooth.

As is known, with the operator $L(t, x)$ there is associated a Markov process $X(t, \omega)$ in E_N , the transition probability density $p(s, x, t, y)$ of which is the Green' s function of the equations:

$$\frac{\partial u}{\partial s} + L(s, x)u = 0; \quad (1)$$

$$\frac{\partial u}{\partial t} = L^*(t, y)u. \quad (2)$$

We shall show that under the condition

$$\sum_{i=1}^N [a_{ii}(t, x) + b_i(t, x)x_i] < -\delta < 0 \quad \text{for } r > r_1 \quad (3)$$

the probability distribution “does not spread out” as time passes. Under additional restrictions on the coefficients it turns out that

$$p(s, x, t, y) - p(s, z, t, y) \rightarrow 0$$

as $t - s \rightarrow \infty$, for arbitrary x and z from E_N .

Theorem 1. *If the coefficients satisfy condition (3), then there exists*

$$\lim_{s \rightarrow -\infty} p(s, x, t, y) = p(t, y) > 0.$$

It is not difficult to show that the function $p(t, y)$, like $p(s, x, t, y)$, satisfies the Chapman-Kolmogorov relation and, consequently, equation (2) for all t ($-\infty < t < \infty$). This function is obviously bounded if the following condition is fulfilled:

A. There exists a number M such that, for all s, x , and y , the inequality

$$p(s, x, s + 1, y) < M$$

holds.

However, even when this condition is fulfilled, equation (2) may have several bounded positive solutions, defined for all t and such that

$$\int_{E_N} p(t, y) dy = 1. \quad (4)$$

For example, the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2} + \frac{\partial}{\partial y}(yu)$$

has the solutions

$$u_1 = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}y^2\right) \quad \text{and} \quad u_2 = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}(y - e^{-t})^2\right\}.$$

It is clear that, in order to single out the unique bounded everywhere solution of equation (2), this solution must be subject to an additional condition.

Theorem 2. Suppose that conditions A^* and (3) are satisfied. Then the function $p(t, y)$ is the unique bounded solution of equation (2) in the domain $-\infty < t < \infty$, for which (4) is satisfied and the integral in (4) converges uniformly.

In this case the process has the ergodic property (see (1)):

Theorem 3. Suppose that conditions A and (3) are satisfied, and let K be an arbitrary compact set in E_N . Then

$$p(s, x, t, y) - p(s, z, t, y) \rightarrow 0,$$

$$\int_{E_N} |p(s, x, t, y) - p(s, z, t, y)| dy \rightarrow 0 \quad (5)$$

as $t \rightarrow \infty$, uniformly with respect to x and z from K . Relations (5) remain valid if the function $p(s, z, t, y)$ is replaced by the function $p(t, y)$.

It is clear that the limiting behavior of the probability distribution density $p(s, x, t, y)$ is closely connected with the asymptotics, for large values of time, of the solutions of the Cauchy problem for equations (1), (2).

Theorem 4. Suppose that conditions A and (3) are satisfied, and let $v(s, x)$ be the solution of the Cauchy problem for $s \leq t$ for equation (1), satisfying the condition

$$v(t, x) = v_0(x). \quad (6)$$

Then

$$\lim_{s \rightarrow -\infty} v(s, x) = \int_{E_N} p(t, y) v_0(y) dy,$$

and the convergence is uniform in each compact set $K \subset E_N$.

Theorem 5. Suppose that conditions A and (3) are satisfied; let $u(t, y)$ be the solution of the Cauchy problem for $t \geq s$ for equation (2) with the initial condition

$$u(s, y) = u_0(y). \quad (7)$$

If

$$\int_{E_N} |u_0(y)| dy < \infty,$$

then

$$u(t, y) - p(t, y) \int_{E_N} u_0(y) dy \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

uniformly in each compact set $K \subset E_N$. If

$$\int_{E_N} u_0(y) dy = \infty,$$

then

$$u(t, y) \xrightarrow[t \rightarrow \infty]{} \infty.$$

The proof of Theorems 1 and 4, which are essentially equivalent, is carried out according to the following plan. Let B be some sphere in E_N , and suppose that condition (3) is satisfied. Denote by $C(t, T)$ the event consisting in the fact that the trajectory $X(t, \omega)$ of the process under consideration visits the set B at least once on the time interval $[t, t + T]$. One can prove the following assertion, fundamental for the proof of Theorem 4 (an analogous lemma was proved by A. N. Kolmogorov in ⁽²⁾ for homogeneous Markov chains with a countable number of states):

Lemma. For any $\varepsilon > 0$ and compact set K there exist a number T , depending only on ε , and a number S , depending only on K , such that

$$\mathbf{P}\{C(t, T)/X(s, \omega) = x\} > 1 - \varepsilon$$

for all $x \in K$, $t > S + s$.

* It is clear that condition A is satisfied, for example, if all the coefficients of equations (1) and (2) are bounded together with their first derivatives.

From this lemma two corollaries follow easily.

1. Let G be an open set in E_N . Then there exists $\delta_G > 0$ such that

$$\mathbf{P}\{X(t, \omega) \in G/X(s, \omega) = x\} > \delta_G,$$

if $x \in K$, $t - s > T(K)$.

2. For every $\varepsilon > 0$ there exists a compact set K_1 such that

$$\mathbf{P}\{X(t, \omega) \in K_1/X(s, \omega) = x\} > 1 - \varepsilon,$$

if $x \in K$, $t > s > T(K)$.

If $v(s, x)$ is a solution of problem (1), (6), then one can choose a sequence $s_k \rightarrow -\infty$ such that the sequence $v(s_k, x)$ tends to $\varphi(x)$ uniformly on every compact set. Relying on corollaries 1 and 2, it is not hard to show that $\varphi(x) \equiv \mu = \text{const}$ and $\mu > 0$, if $v_0(x) \geq 0$ and $v_0(x) \not\equiv 0$. Consequently, $v(s, x) \rightarrow \mu$ as $s \rightarrow -\infty$.

Studying further the solutions of problems (1), (6) and (2), (7) and using Green's formula for equations (1), (2), one can determine how this limiting value μ is connected with the solution of equation (2), and prove Theorems 2, 3, and 5.

Let us note that condition (3) is fairly sharp, since when the condition

$$\sum_{i=1}^N (a_{ii} + b_i x_i) > \delta > 0$$

is satisfied for $r > r_1$, the distribution density $p(s, x, t, y) \rightarrow 0$ as $s \rightarrow -\infty$ (3). If the process $X(t, \omega)$ is considered on a compact manifold M , then the conclusions of Theorems 1-5 are valid under minimal restrictions on the coefficients. It suffices, for example, to assume that all coefficients are continuous, bounded, and

$$\sum_{i,j=1}^N a_{ij} \xi_i \xi_j > m \sum_{i=1}^N \xi_i^2$$

for all real ξ_i and $(t, x) \in (-\infty, \infty) \times M$.

The results obtained also extend to the case of a bounded domain in E_N , if on the boundary of this domain the solution of equation (1) satisfies the condition $\partial u / \partial \nu = 0$ (ν is the conormal to the boundary of the domain), while the solution of equation (2) satisfies the adjoint boundary condition.

In the case of homogeneous Markov processes ($L = \tilde{L}(x)$), questions analogous to those considered in the present note were studied, under more general assumptions, in (4). Let us note in conclusion that a theorem similar to Theorem 4 was proved earlier for the one-dimensional case by Ya. I. Kanel (5).

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Note: Figure translations are in progress. See original paper for figures.

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