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1962

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Abstract

Full Text

MATHEMATICS

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ON COMPUTABLE INVARIANTS

1. In considering a reflexive, symmetric, and transitive relation $\mathfrak{R}$, defined for constructive objects of a given kind, one may introduce the concept of a **computable invariant** of this relation. By this we mean an algorithm applicable to every object of this kind and transforming any two objects connected by the relation $\mathfrak{R}$ into one and the same object. We say of an algorithm applicable to every object of the kind under consideration that it is a **computable invariant of the relation $\mathfrak{R}$ for the object** P if this algorithm transforms into one and the same object all objects connected with P by the relation $\mathfrak{R}$. We say that the objects P and Q are **indistinguishable by invariants of the relation** $\mathfrak{R}$ if $I(P) = I(Q)$ whenever I is a computable invariant of the relation $\mathfrak{R}$ both for P and for Q.

In what follows, the constructive objects are words in a given alphabet satisfying a certain condition. The term "algorithm," in accordance with the principle of normalization (Church's thesis), is understood as a "normalizable algorithm." By an "invariant" we mean a computable invariant. The terminology and notation of the monograph ([2]) are used, but defining systems of associative calculi are regarded as ordered systems of relations.

2. Let P and Q be words in the alphabet A; let V and W be relations in this alphabet. We say that V is **adjacent to W according to the relation** $P \leftrightarrow Q$ if there exist words R and S (in the alphabet $A \cup \{\leftrightarrow\}$) such that $V = RPS$, while $W = RQS$, or such that $V = RQS$, while $W = RPS$.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be associative calculi in one and the same alphabet. Let $\{V_i \ (i = 1, \dots, m)\}$ and $\{W_i \ (i = 1, \dots, n)\}$ be, respectively, their defining systems. We say that $\mathfrak{A}$ is **directly transformable into** $\mathfrak{B}$ if $m = n$ and there are natural numbers i and j satisfying the conditions: $1 \leq i \leq n$, $1 \leq j \leq n$, $i \neq j$, $V_k = W_k$ for $1 \leq k \leq n$ and $k \neq i$, and V_i is adjacent to W_i according to V_j.

We say of associative calculi $\mathfrak{A}$ and $\mathfrak{B}$ that they are **mutually transformable** if there is a sequence of associative calculi $\mathfrak{A}_0, \dots, \mathfrak{A}_l$ $(l \geq 0)$ such that $\mathfrak{A}_0$ is $\mathfrak{A}$, $\mathfrak{A}_l$ is $\mathfrak{B}$, and $\mathfrak{A}_{r-1}$ is directly transformable into $\mathfrak{A}_r$ for $0 < r \leq l$. We say of associative calculi $\mathfrak{A}$ and $\mathfrak{B}$ in one and the same alphabet that they are **equivalent** if the same word equivalences occur in them.

Any two mutually transformable calculi are equivalent, and any two equivalent calculi are isomorphic. Mutual transformability, equivalence, and isomorphism of calculi in a given alphabet are reflexive, symmetric, and transitive. In considering invariants of these relations, we identify associative calculi with their records and treat the invariants as algorithms applied to records of associative calculi.

We say of an associative calculus in the alphabet $\{\xi_1, \dots, \xi_q\}$ that it is **trivial** if the first p relations of its defining"】【

systems are $\Lambda \leftrightarrow \Lambda$, and the remaining q are relations $\xi_i \leftrightarrow \Lambda$ $(i = 1, \dots, q)$. Here p may be any natural number.

Every trivial calculus is unitary.

3. A special kind of associative calculi is formed by **inverse calculi**, which are constructed as follows. The alphabet of the calculus consists of two parts—the **positive alphabet** A and the **negative alphabet** A^{-1}. A one-to-one correspondence is established between them. The letter of the alphabet A^{-1} (A) corresponding to a letter ξ of the alphabet A (A^{-1}) is denoted by the symbol ξ^{-1}. At the end of the defining system there occur all relations of the form $\xi\xi^{-1} \leftrightarrow \Lambda$, where ξ is a letter of the alphabet of the calculus $A \cup A^{-1}$. All other relations also have an empty right-hand side.

When specifying an inverse calculus in a $2q$-letter alphabet $A \cup A^{-1}$, we shall omit the $2q$ relations of the form $\xi\xi^{-1} \leftrightarrow \Lambda$ standing at the end of the defining system. The remaining system of relations will be called the **reduced defining system** of the inverse calculus under consideration. It has the form $\{R_i \leftrightarrow \Lambda\ (i = 1, \dots, m)\}$, where R_i are words in the alphabet $A \cup A^{-1}$.

Let $\mathfrak{G}$ and $\mathfrak{H}$ be inverse calculi in one and the same alphabet Γ^* with reduced defining systems $\{R_i \leftrightarrow \Lambda\ (i = 1, \dots, m)\}$ and, respectively, $\{S_i \leftrightarrow \Lambda\ (i = 1, \dots, n)\}$. We say that $\mathfrak{G}$ **passes** into $\mathfrak{H}$ if $m = n$ and one of the following conditions is satisfied:

1°. There exist a natural number i, a letter ξ of the alphabet Γ, and a word U in Γ such that $1 \le i \le n$, $R_k = S_k$ for $1 \le k \le n$ and $k \ne i$, $R_i = \xi U$, $S_i = U\xi$.

2°. There exist natural numbers i and j such that $1 \le i \le n$, $1 \le j \le n$, $i \ne j$, $R_k = S_k$ for $1 \le k \le n$ and $k \ne i$, $R_{iR}j = S_i$.

3°. There exist a natural number i and a letter ξ of the alphabet Γ such that $1 \le i \le n$, $R_k = S_k$ for $1 \le k \le n$ and $k \ne i$, $R_i\xi\xi^{-1} = S_i$.

We say of inverse calculi $\mathfrak{G}$ and $\mathfrak{H}$ that they are **mutually transformable** if there is a sequence of inverse calculi $\mathfrak{G}_0, \dots, \mathfrak{G}_l$ $(l \ge 0)$ such that $\mathfrak{G}_0$ is $\mathfrak{G}$, $\mathfrak{G}_l$ is $\mathfrak{H}$, and for $0 < r \le l$, $\mathfrak{G}_{r-1}$ passes into $\mathfrak{G}_r$, or $\mathfrak{G}_r$ into $\mathfrak{G}_{r-1}$.

Any two mutually transformable inverse calculi are equivalent. Mutual transformability of inverse calculi in a given alphabet is reflexive, symmetric, and transitive. The invariants of mutual transformability, equivalence, and isomorphism of inverse calculi in a given alphabet we regard as algorithms applied to their "reduced notations." By the **reduced notation** of an inverse calculus with reduced defining system $\{R_i \leftrightarrow \Lambda \ (i = 1, \ldots, m)\}$ we here mean the word $R_1\alpha \ldots R_m\alpha$, where α is a fixed letter not belonging to the alphabet of the calculus.

We say of an inverse calculus with positive alphabet $\{\xi_1, \ldots, \xi_q\}$ that it is **manifestly unitary** if the first p relations of its reduced defining system have the form $\Lambda \leftrightarrow \Lambda$, and the remaining q are relations $\xi_i \leftrightarrow \Lambda \ (i = 1, \ldots, q)$. Here p may be any natural number.

Every manifestly unitary calculus is unitary.

4. **Theorem 1.** *An inverse calculus $\mathfrak{G}$ and a word A in its alphabet can be constructed in such a way that the following conditions are satisfied:*

1°. *A is not equivalent to the empty word in $\mathfrak{G}$.*

2°. *A is indistinguishable from the empty word by invariants of equivalence of words in $\mathfrak{G}$.*

Theorem 2. *Whatever associative calculus $\mathfrak{A}$ may be, an associative calculus $\mathfrak{B}$ can be constructed in the alphabet $\{a, b, c, d\}$, satisfying the conditions:*

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* This means that the calculi $\mathfrak{G}$ and $\mathfrak{H}$ have the same positive alphabets, the same negative alphabets, and that the correspondence between the positive and negative alphabets is the same.

1°. $\mathfrak{A}$ is embeddable in $\mathfrak{B}$.

2°. $\mathfrak{B}$ is indistinguishable, with respect to the invariants of mutual transformability, from the trivial calculus in the alphabet $\{a, b, c, d\}$.

Theorem 3. Whatever inverse calculus $\mathfrak{G}$ may be, an inverse calculus $\mathfrak{H}$ with positive alphabet $\{y, z, x, t, s, a, b, d\}$ can be constructed, satisfying the conditions:

1°. $\mathfrak{G}$ is embeddable in $\mathfrak{H}$.

2°. $\mathfrak{H}$ is indistinguishable, with respect to the invariants of mutual translatability, from the explicitly unitary calculus with positive alphabet $\{y, z, x, t, s, a, b, d\}$.

Theorem 4. Four-dimensional polyhedra M and N can be constructed, satisfying the conditions:

1°. M is simply connected.

2°. N is not simply connected.

3°. M and N are indistinguishable with respect to the invariants of combinatorial equivalence of 4-dimensional polyhedra.

5. Theorem 1 is a strengthening of the well-known theorem of P. S. Novikov on the possibility of constructing an inverse calculus with an undecidable word equivalence problem ([7]). Indeed, in the inverse calculus $\mathfrak{G}$ constructed according to Theorem 1, the word equivalence problem cannot be decidable.

From Theorem 2 we obtain

Corollary 1. Whatever associative calculi $\mathfrak{A}$ and $\mathfrak{C}$ may be, associative calculi $\mathfrak{D}$ and $\mathfrak{F}$ can be constructed in one and the same alphabet, satisfying the conditions:

1°. $\mathfrak{A}$ is embeddable in $\mathfrak{D}$.

2°. $\mathfrak{F}$ is isomorphic to $\mathfrak{C}$.

3°. $\mathfrak{D}$ and $\mathfrak{F}$ are indistinguishable with respect to the invariants of mutual transformability.

If, moreover, the alphabet of the calculus $\mathfrak{C}$ contains q letters, then $\mathfrak{D}$ and $\mathfrak{F}$ can be constructed as associative calculi in a $(q+4)$-letter alphabet.

This is a strengthening of the theorem on the impossibility of algorithms recognizing certain properties of associative calculi (see ([3]) or ([2]), p. 345).

From Theorem 3 it follows

Corollary 2. Whatever inverse calculi $\mathfrak{G}$ and $\mathfrak{K}$ may be, inverse calculi $\mathfrak{M}$ and $\mathfrak{N}$ can be constructed in one and the same alphabet, satisfying the conditions:

1°. $\mathfrak{G}$ is embeddable in $\mathfrak{M}$.

2°. $\mathfrak{N}$ is isomorphic to $\mathfrak{K}$.

3°. $\mathfrak{M}$ and $\mathfrak{N}$ are indistinguishable with respect to the invariants of mutual translatability.

If, moreover, the positive alphabet of the calculus $\mathfrak{K}$ contains q letters, then $\mathfrak{M}$ and $\mathfrak{N}$ can be constructed as inverse calculi with a $(q+8)$-letter positive alphabet.

This is a strengthening of the theorem of Adian–Rabin ([1,10]) on the impossibility of algorithms recognizing certain properties of inverse calculi.

Theorem 4 has

Corollary 3. For every natural number n greater than three, n-dimensional polyhedra M^n and N^n can be constructed, satisfying the conditions:

1°. The fundamental groups of the polyhedra M^n and N^n are nonisomorphic.

2°. M^n and N^n are indistinguishable with respect to the invariants of combinatorial equivalence of n-dimensional polyhedra.

Hence follows the main result of note (5). Indeed, having constructed M^n and N^n according to Corollary 3, we easily convince ourselves that, for any binary relation $\mathfrak{R}$ lying between combinatorial equivalence and "kinship," the problem of recognizing the existence of the relation $\mathfrak{R}$ to M^n among n-dimensional polyhedra is undecidable.

6. At the basis of all these results lies a well-known theorem on the existence of a partially recursive function taking only two values

and not extendable to a general recursive function. Translated into the language of normal algorithms, this theorem may be formulated as follows.

A normal algorithm $\mathfrak{A}$ over the alphabet $\{a, b\}$ can be constructed, satisfying the following conditions:

1°. $\mathfrak{A}(P) = a$, or $\mathfrak{A}(P) = b$, whenever the algorithm $\mathfrak{A}$ is applicable to the word P in the alphabet $\{a, b\}$.

2°. There is no normal algorithm $\mathfrak{B}$ over the alphabet $\{a, b\}$, applicable to every word in this alphabet and such that $\mathfrak{A}(P) = \mathfrak{B}(P)$, whenever the algorithm $\mathfrak{A}$ is applicable to the word P in the alphabet $\{a, b\}$.

Applying to the algorithm $\mathfrak{A}$, constructed according to this theorem, the theorem on the representability of a normal algorithm by means of an associative calculus (see (6) or (2), p. 208), and using the theorem of P. S. Novikov and S. I. Adian on the representability of an associative calculus by means of an inverse calculus (see (8), p. 86), we obtain Theorem 1. Theorem 2 follows from this with the aid of the construction used in the proof of the theorem on the undecidability of properties of associative calculi (see (3) or (2), pp. 345–362). Theorem 3 is proved in an analogous way with the aid of Theorem 1 and the construction used by Rabin (10). In this, one also uses the theorem of Higman, Neumann, and Neumann on the embedding of inverse calculi in inverse calculi with a two-letter positive alphabet (9). Finally, Theorem 4 can be proved with the aid of Theorem 3 and the construction outlined in notes (4,5).

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Received
30 VI 1962

CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.