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Abstract

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MATHEMATICS

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ON SLIDING OPTIMAL REGIMES*

(Presented by Academician L. S. Pontryagin, 15 XII 1961)

The maximum principle in the theory of optimal processes (see ([1])) gives necessary conditions satisfied by optimal controls and trajectories. These conditions are sufficient for finding an optimal process, if it exists. The most general existence theorem for optimal processes, due to A. F. Filippov ([2]), imposes a very strong condition on the right-hand side of equation (1), which, as a rule, is not fulfilled for a function f nonlinear with respect to u.

If the condition of Filippov' s theorem is violated, then, generally speaking, an optimal control does not exist. However, there almost always exists such a sequence of admissible controls that the corresponding sequence of trajectories of equation (1) converges to some limiting curve satisfying the prescribed boundary conditions, while the values of the functional being minimized tend to their lower bound. Along the limiting curve there is naturally defined a motion of the phase point, although this motion is not necessarily realizable as motion along a trajectory of equation (1) corresponding to some control. However, this "limiting motion" can be approximated arbitrarily accurately by motion along a trajectory of equation (1). If the limiting curve is not a trajectory of equation (1), then we shall call the motion along it a **sliding optimal regime**.

Thus, under fairly broad assumptions, formulated precisely below, the optimal problem always has a solution, realized either in the form of an ordinary optimal control, or in the form of an optimal sliding regime.

All optimal solutions of the first kind are found with the aid of the maximum principle. V. F. Krotov first found optimal sliding regimes for one concrete second-order system ([3], [4]). In the present note a general method is set forth for finding optimal sliding regimes, which reduces the determination of such regimes to the maximum principle.

For simplicity we consider the problem of optimal speed. The extension of the results obtained to the case in which an arbitrary integral functional is minimized presents no difficulty.

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Let X^n be the n-dimensional phase space of the point depicted by $x = (x^1, \dots, x^n)$; let u be an r-dimensional control vector; and let the domain Ω of admissible values of the vector u be a compact, connected set of r-dimensional space. The motion of the point x is governed by the equation

$$\dot{x} = f(x, u), \tag{1}$$

where the function f is continuous in the totality of its arguments and continuously differentiable with respect to x. By a trajectory of equation (1) we shall mean any absolutely continuous solution $x(t)$, $t_0 \le t \le t_1$, of the equation, correspond-

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* The work was carried out in L. S. Pontryagin' s seminar on the theory of oscillations and automatic control.

corresponding to some control, i.e., to some measurable function $u(t)$, $t_0 \le t \le t_1$, with values in Ω.

Suppose that the points $a, b \in X^n$ can be joined by a trajectory of equation (1). In this case one can choose a minimizing sequence of controls $u^{(k)}(t)$, $0 \le t \le t_k$, $k = 1, 2, \dots$, which transfer the phase point from a to b along the trajectories $x_{(k)}(t)$, $0 \le t \le t_k$, and minimize the transition time: $\lim t_k = T$, where T is the exact lower bound of the transition times from a to b along trajectories of equation (1). We shall assume that all the trajectories $x_{(k)}(t)$, $0 \le t \le t_k$, $k = 1, 2, \dots$, are bounded in modulus by one and the same constant. This assumption is satisfied if, for example, the estimate $x \cdot f(x, u) \le c|x|^2$ holds. Other a priori estimates can also be given under which our assumption is valid.

Thus, we may assume that the sequence of trajectories $x_{(k)}(t)$, $0 \le t \le t_k$, converges uniformly to some limiting curve $x(t)$, $0 \le t \le T$. This curve is an optimal trajectory of equation (1) joining the points a, b, if motion along it can be effected by means of some control; otherwise the motion specified along $x(t)$ is a sliding optimal regime for equation (1), transferring the phase point from a to b.

Our problem consists in finding a minimizing sequence of controls $u^{(k)}(t)$, $0 \le t \le t_k$, $k = 1, 2, \dots$.

Let us consider, instead of equation (1), the controlled equation

$$\dot{x} = \sum_{\alpha=1}^{n} p_\alpha f(x, u_\alpha) = g(x, p, U), \tag{2}$$

in which the control vector is the vector $(p, U) = (p_1, \dots, p_n, u_1, \dots, u_n) \in T^n \cdot \Omega^n$, where the point $p = (p_1, \dots, p_n)$ belongs to the n-dimensional simplex

$$T^n = \{p = (p_1, \dots, p_n) : p_1 + \dots + p_n = 1,\ p_\alpha \ge 0,\ \alpha = 1, \dots, n\},$$

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and the point $U = (u_1, \dots, u_n)$ belongs to the n-th topological power Ω^n of the set Ω. Thus, a control for equation (2) is any measurable function with values in $T^n \cdot \Omega^n$.

For each fixed x, the image $G(x)$ of the set $T^n \cdot \Omega^n$ under the mapping $g(x, p, U) : T^n \cdot \Omega^n \to X^n$ is convex in X^n. Indeed, let $F(x)$ denote the image of the set Ω under the mapping $f(x, u) : \Omega \to X^n$. It follows from (2) that $G(x)$ coincides with the union of the convex hulls of arbitrary n points (not necessarily distinct) taken from $F(x)$. Since the set $F(x)$ is connected (as the continuous image of the connected set Ω), $G(x)$ coincides with the convex hull of the set $F(x)$.

Consequently, by the existence theorem (see ([2])), any two points $a, b \in X^n$ that can be joined by some trajectory of equation (2) can also be joined by an optimal trajectory with optimal transition time T. Obviously, it is impossible to get from a to b along a trajectory of equation (1) in less than time T. It turns out, however, that in time T one can get from a to b either along an optimal trajectory of equation (1), or by means of a sliding optimal regime.

Moreover, we shall show how, with the aid of an arbitrary control $(p(t), U(t)) = (p_1(t), \dots, p_n(t), u_1(t), \dots, u_n(t))$, $0 \le t \le t^*$, for equation (2), which transfers the phase point from a to b along the trajectory $x(t)$, $0 \le t \le t^*$, one can construct a sequence of controls $u^{(k)}(t)$, $0 \le t \le t^*$, $k = 1, 2, \dots$, for equation (1), to which correspond trajectories $x_{(k)}(t)$, $0 \le t \le t^*$, $x_{(k)}(0) = a$, converging uniformly to the trajectory $x(t)$, $0 \le t \le t^*$, of equation (2).

Subdivide the interval $0 \le t \le t^*$ into k partial intervals $I_i^{(k)}$, $i = 1, \dots, k$. Divide the interval $I_i^{(k)}$ into n measurable nonintersecting sets $\mathcal{E}_{i\alpha}^{(k)}$, $\alpha = 1, \dots, n$, $I_i^{(k)} = \bigcup_{\alpha=1}^{n} \mathcal{E}_{i\alpha}^{(k)}$, satisfying the single condition-

i.e., $\operatorname{mes} \mathcal{E}_{i\alpha}^{(k)} = \displaystyle\int p_\alpha(t)\, dt$. Define the control $u^{(k)}(t)$, $0 \le t \le t^*$, $I_i^{(k)}$, by means of the relation $u^{(k)}(t) = u_\alpha(t)$ for $t \in \mathcal{E}_{i\alpha}^{(k)}$, $\alpha = 1, \dots, n$; $i = 1, \dots, k$. If, as $k \to \infty$, the greatest of the lengths of the intervals $I_i^{(k)}$ tends to zero, then the sequence $u^{(k)}(t)$, $0 \le t \le t^*$, will be the desired one.

We shall call the functions $u_\alpha(t)$, $0 \le t \le t^*$, $\alpha = 1, \dots, n$, **basic controls**; and $p_\alpha(t)$, $0 \le t \le t^*$, $\alpha = 1, \dots, n$, **weight functions**. From the described construction of the controls $u^{(k)}(t)$, $0 \le t \le t^*$, it follows that in the limit, as $k \to \infty$, the point $u^{(k)}(t)$ may be regarded as "jumping infinitely often" from one basic control to another and remaining on the basic control $u_\alpha(t)$ for a fraction of time whose density is given by the weight function p_α; the motion of the phase point $x_{(k)}(t)$, in the limit, is a "sliding" along the absolutely continuous trajectory $x(t)$, $0 \le t \le t^*$, of equation (2).

Thus, the problem of finding an optimal sliding regime for equation (1), taking the phase point from a to b, is reduced to finding an optimal control for equation (2), taking the phase point from a to b. Therefore one must construct (see ([1])) the system of equations adjoint to (2) for the coordinates of the vector

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$\psi = (\psi_1, \dots, \psi_n)$:

$$\dot{\psi}_i = -\sum_{\alpha=1}^{n} p_\alpha \frac{\partial f(x, u_\alpha)}{\partial x^i} \cdot \psi = -\frac{\partial g(x, p, U)}{\partial x^i} \cdot \psi \tag{3}$$

and find all solutions $\psi(t), x(t), p(t), U(t)$, $0 \le t \le t^*$, of the system (2)–(3), satisfying the boundary conditions $x(0) = a$, $x(t^*) = b$, and the maximum condition

$$\mathcal{H}(\psi(t), x(t), p(t), U(t)) = \mathcal{M}(\psi(t), x(t)) \ge 0, \qquad 0 \le t \le T,$$

where

$$\mathcal{H}(\psi, x, p, U) = \psi \cdot g(x, p, U), \qquad \mathcal{M}(\psi, x) = \sup_{(p,U) \in T^n \cdot \Omega^n} \mathcal{H}(\psi, x, p, U).$$

Introduce the notation $H(\psi, x, u) = \psi \cdot f(x, u)$ and rewrite the maximum condition in the form

$$\sum_{\alpha=1}^{n} p_\alpha(t)\, H(\psi(t), x(t), u_\alpha(t)) = \mathcal{M}(\psi(t), x(t)) \ge 0,$$

from which we conclude that all the basic controls $u_\alpha(t)$, $0 \le t \le t^*$, $\alpha = 1, \dots, n$, satisfy one and the same equation

$$H(\psi(t), x(t), u_\alpha(t)) = \sup_{u \in \Omega} H(\psi(t), x(t), u), \qquad 0 \le t \le t^*. \tag{4}$$

Obviously, if this equation has a unique solution for almost all t, $0 \le t \le t^*$, then the curve $x(t)$, $0 \le t \le t^*$, is a trajectory of equation (1). Thus, a sliding optimal regime can occur only in the case when the corresponding equation (4), with respect to u_α, has at least two solutions on a set of positive measure on the time axis.

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Note: Figure translations are in progress. See original paper for figures.

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