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Abstract

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ON THE THEORY OF CURVES IN PROJECTIVE SPACE OF n DIMENSIONS

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In the present paper a natural frame and differential invariants of the least differential order are constructed for a curve in projective space P_n of n dimensions, and an application is given of the results obtained to linear differential equations.

1. We shall interpret a point in P_n as a central ray in the central-affine $(n+1)$-dimensional space E_{n+1}. To a curve C in P_n there will correspond in E_{n+1} a two-dimensional central conical surface S with directrix K, whose equation is $\mathbf{r} = \mathbf{r}(t)$, and the admissible transformation of the directrix

$$\tilde{\mathbf{r}} = \varphi^{-1}\mathbf{r},$$

where the function $\varphi = \varphi(t)$, differentiable a sufficient number of times and nonzero, is at the same time an admissible transformation of the homogeneous coordinates of the point in P_n.

Since every generator of the surface S is a central-affine one-dimensional space E_1, with the curve C there is associated a fibered space $E_1(K)$. We shall assume that in this space a certain field of local coordinate systems has been chosen and that, under the transformation of the directrix $\tilde{\mathbf{r}} = \varphi^{-1}\mathbf{r}$, the field of local coordinate systems is transformed as follows: $x^* = \varphi^{-1}x$. Since, moreover, the curve K is parametrized and, consequently, so is X_1, there is naturally associated with it the holonomized tangent fibered space, which, for convenience of reference, we shall denote by $T_1(X_1)$. Therefore with the curve C in P_n there is associated the doubled fibered space* $E_1 \times T_1(X_1)$ (^{3}).

If the curve C does not lie in a plane of dimension less than n, then among the vectors

$$\mathbf{r}^k = \frac{1}{k!}\frac{d^k\mathbf{r}}{dt^k}, \quad k = 0, 1, \dots,$$

the first $n+1$ are linearly independent, and there is a unique relation

$$\mathbf{r}^{n+1} + \sigma_1\mathbf{r}^n + \dots + \sigma_n\mathbf{r}^1 + \sigma_{n+1}\mathbf{r}^0 = 0. \tag{1}$$

We shall assume that, under transformations of coordinates in E_1 and in T_1, the coefficients σ are transformed in such a way that equation (1) remains invariant. Then the connecting object σ_k $(k = 1, 2, \dots, n+1)$ determines the curve C up to automorphisms of the space P_n.

Let in $E_1 \times T_1(X_1)$ a geometric differential object Ω_a $(a, b = 1, 2, \dots, N)$ be given with transformation laws

$$\widetilde{\Omega}_a = \Phi_a(\Omega_b, \varphi_k);$$
$$^*\Omega_a = F_a(\Omega_b, f_k)$$

under transformations of coordinates in E_1 and in T_1, respectively, where

$$\varphi_k = \frac{1}{k!}\frac{d^k\varphi}{dt^k}, \quad f_k = \frac{1}{k!}\frac{d^k f}{dt^k}, \quad k = 0, 1, \dots .$$

Introduce the operators

$$D_m\Omega_a = \left(-\frac{\partial\Phi_a(\Omega_b, \varphi_k)}{\partial\varphi_m}\right)_{\varphi_k=\delta_k^0}, \quad L_m\Omega_a = \left(-\frac{\partial F_a(\Omega_b, f_k)}{\partial f_m}\right)^{(2)}_{f_k=\delta_k^1},$$

for which we obtain the recurrence relations

$$D_m\frac{d^k\Omega_a}{dt^k} = \frac{d}{dt}D_m\frac{d^{k-1}\Omega_a}{dt^{k-1}} + mD_{m-1}\frac{d^{k-1}\Omega_a}{dt^{k-1}},$$
$$L_m\frac{d^k\Omega_a}{dt^k} = \delta_m^1\frac{d^k\Omega_a}{dt^k} + \frac{d}{dt}L_m\frac{d^{k-1}\Omega_a}{dt^{k-1}} + mL_{m-1}\frac{d^{k-1}\Omega_a}{dt^{k-1}}, \tag{2}$$

where δ_k^s are the Kronecker symbols.

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* We use the term "doubled fibered space" instead of "doubled composite manifold" (see (^{2}), p. 98).

An object W is then and only then a density of weight (k, p) (i.e., a relative invariant of weight k in E_1 and weight p in T_1) when (see (2))

$$D_mW = k\delta_m^0 W; \qquad L_mW = p\delta_m^1 W. \tag{3}$$

2. Using the known differentiation formulas or, successively applying relations (2), we obtain $D_m{}^k\mathbf{r} = {}^{k-m}\mathbf{r}$, $L_m{}^k\mathbf{r} = (k-m+1)^{k-m+1}\mathbf{r}$, and everywhere it is to be assumed that an object with a negative index is zero (for example, ${}^{-s}\mathbf{r} = 0$). Applying condition (3) to the left-hand side of equation (1), we obtain

$$D_0\sigma_k = 0;\quad D_m\sigma_k = -\sigma_{k-m},\quad m > 0;\quad k = 1, 2, \dots, n+1\quad (\sigma_0 = 1); \tag{4a}$$

$$L_1\sigma_k = k\sigma_k;\quad L_m\sigma_k = -(n-k+1)\sigma_{k-m+1},\quad m > 1;\quad k = 1, 2, \dots, n+1\quad (\sigma_0 = 1). \tag{4b}$$

The vectors

$$\mathbf{r}_k = \sum_{s=0}^{k}\sigma_{k-s}\,{}^{s}\mathbf{r}\quad\text{and}\quad \widetilde{\nabla}\mathbf{r}_k = \frac{d}{dt}\mathbf{r}_k + \sigma_1\mathbf{r}_k,\quad k = 0, 1, \dots, n,$$

form a connecting object with vector components in $E_1 \times T_1(X_1)$, which is a density of weight 1 in E_1, and the first of them are linearly independent. Then there is a unique decomposition

$$\widetilde{\nabla}\mathbf{r}_0 = \mathbf{r}_1;\qquad \widetilde{\nabla}\mathbf{r}_k = (k+1)\mathbf{r}_{k+1} + \sum_{s=0}^{k} I_{k-s+1}\mathbf{r}_s,\quad k = 1, 2, \dots, n, \tag{5}$$

where one should set $\mathbf{r}_{n+1} = 0$, and the coefficients I_k, called the projective invariants of the curve, are equal to

$$\begin{aligned} I_1 = 0;\qquad I_{k+1} = {} & \sum_{h=1}^{k}\frac{d}{dt}\sigma_{k-h}\sum^{(h)}(-1)^p\frac{p!}{i_1!i_2!\cdots i_h!}\sigma_1^{i_1}\sigma_2^{i_2}\cdots\sigma_h^{i_h} \\ & + (k+1)\sum^{(k+1)}(-1)^p\frac{(p-1)!}{i_1!i_2!\cdots i_{k+1}!}\sigma_1^{i_1}\sigma_2^{i_2}\cdots\sigma_{k+1}^{i_{k+1}}. \end{aligned} \tag{6}$$

($\sum^{(h)}$ everywhere denotes summation over all ordered systems of nonnegative integers $i_1, i_2, \dots, i_h$ satisfying the equation $i_1 + 2i_2 + \dots + hi_h = h$, and the number p under the summation sign is to be taken equal to $p = i_1 + i_2 + \dots + i_h$.) The projective invariants I_k $(2 \le k \le n+1)$ coincide with the projective invariants of N. F. Rzhekhina (^{1}), have order $n+2$ (the smallest possible), and, together with σ_1, determine the curve up to automorphisms of the space P_n. Denoting

$$b_k = \frac{1}{n+2}I_k,\quad 2 \le k \le n+1;\qquad b_0 = 1,$$

according to (2) and (4) we have

$$L_1 b_k = k b_k;\qquad L_m b_k = (k-m)b_{k-m+1},\quad m > 1;\quad 2 \le k \le n+1. \tag{7}$$

It follows that the collection of quantities $b_0, b_2, b_3, \dots, b_{n+1}$ is an object in T_1.

3. The objects

$$G_{k-3} = \frac{d}{dt} b_{k-1} - (k+1) b_k - \sideset{}{'}\sum_{i+j=k} b_i b_j, \quad k = 3, 4, \dots, n+1, \tag{8}$$

($\sum'$ denotes summation over all positive integers i, j greater than 1 and such that $i + j = k$) satisfy the condition: if $G_s = 0$ ($s = 0, 1, \dots, h-1$), then G_k ($k = 0, 1, \dots, h-1, h$) are densities of weight $(0, k+3)$, respectively.

The maximal number q such that $G_0 = G_1 = \dots = G_{q-1} = 0$ is called the class of the curve C and is an important arithmetic invariant of it, which can take values from 0 to $n-1$ (for $q = n-1$ one should take $G_0 \neq 0$).

a) If $q < n-2$, then $\dot{\gamma} = 2G_{q+1}/G_q$ is an object of affine connection in $T_1(X_1)$; b) if $q = n-2$, then γ is determined by the condition $\nabla G_{n-2} = 0$. In both cases $\Gamma = \sigma_1 + \frac{n}{2}\gamma$ is an object of affine connection in $E_1(X_1)$. Finally, c) if $q = n-1$, then one can introduce such fields of local coordinate systems in $E_1 \times T_1(X_1)$ that the points of the curve C will have coordinates $(1, t, t^2, \dots, t^n)$.

Assuming further that $q < n-1$, and considering the object $b_1 = -2\gamma$, satisfying condition (7) for $k = 1$, we obtain that the vectors

$$\mathbf{R}_k = \mathbf{r}_k + G_1^k \mathbf{r}_{k-1} + \dots + G_{k-1}^k \mathbf{r}_1 + G_k^k \mathbf{r}_0,$$

where

$$G_h^k = \sum^{(h)} \frac{(k-n-1)(k-n-2)\cdots(k-n-p)}{i_1! i_2! \cdots i_h!} b_1^{i_1} b_2^{i_2} \cdots b_h^{i_h}, \tag{9}$$

are densities of weight $(1, k)$, respectively, and therefore form a natural frame of the curve C. It is shown that the order of the natural frame $n+3$ for $q < n-2$ and $n+4$ for $q = n-2$ is minimal; if $q = n-1$, the objects G_k^h cannot be constructed.

4. Having objects of affine connections in $E_1 \times T_1(X_1)$, one can construct the basic differentiation of densities of weight (k, p) (^{3}). Then from the unique expansion

$$D\mathbf{R}_n + \sum_{k=0}^{n} v_{n-k+1} \mathbf{R}_n = 0 \tag{10}$$

there are determined densities v_k of weight $(0, k)$, $k = 1, 2, \dots, n+1$, which have the form

$$v_0 = 0; \qquad v_2 = \frac{d}{dt} b_1 - 3b_2 + b_1^2;$$

$$v_{k+3} = \sum_{h=0}^{k} (k-h+1) G_{k-h} \sum^{(h)} \frac{k(k-1)\cdots(k-p+2)}{i_1! i_2! \cdots i_h!} b_1^{i_1} b_2^{i_2} \cdots b_h^{i_h}, \tag{11}$$

where the objects G_k are found by formula (8), and the densities v_k $(k > 2)$ have minimal order $n+3$, while v_2 has order $n+4$ for $q < n-2$ and order $n+5$ for $q = n-2$.

Let us differentiate each vector of the natural frame basically and expand these derivatives in the vectors of the same frame. We obtain

$$D\mathbf{R}_0 = \mathbf{R}_1; \qquad D\mathbf{R}_k = (k+1)\mathbf{R}_{k+1} + (n-k+1)\sum_{s=0}^{k-1} v_{k-s+1}\mathbf{R}_s, \tag{12}$$

$$k = 1, 2, \dots, n-1.$$

These relations together with (10) constitute the Frenet formulas for the curve C, from which it follows that the densities $v_2, v_3, \dots, v_{n+1}$, together with the objects of affine connections γ and Γ, determine the curve up to automorphisms of the space P_n.

Since $v_q = G_q$ is different from zero, one can introduce the projective arc length of the curve

$$s = \int |v_q|^{1/q}\, dt$$

and construct its invariants (curvatures) of minimal order

$$\varkappa_1 = |v_q|^{-(q+1)/q} \cdot Dv_q; \qquad \varkappa_k = v_k \cdot |v_q|^{-k/q},$$

$$k = 2, 3, \dots, q-1, \quad q+1, \dots, n+1; \tag{13}$$

then from the Frenet formulas it follows that the specification of the invariants $\varkappa_k$ as functions of arc length determines the curve up to automorphisms of the space P_n, for one can always choose φ so that $\tilde{\Gamma} = 0$.

5. Let a homogeneous linear differential equation be given,

$$y^{(n+1)} + a_1 y^{(n)} + \dots + a'_{ny} + a_{n+1} y = 0. \tag{14}$$

It preserves its form under a change of variable and a linear change of the function $y = \varphi\tilde{y}$. With equation (14) there is associated, one-to-one, a curve C in projective n-dimensional space P_n, defined by the object

$$\sigma_k = \frac{(n-k+1)!}{(n+1)!}\, a_k, \quad 1 \le k \le n+1,$$

up to automorphisms of the space P_n. Therefore all objects invariantly connected with the curve C will be objects invariantly connected with the equation; in particular, I_k are called projective invariants, $\varkappa_k$ invariants, and the number q the class of the equation. Then the equation is reduced to the form

$$y^{(n+1)} + g_{q+3}y^{(n-q-2)} + \cdots + g'_{ny} + g_{n+1}y = 0,$$

where the parameter z is chosen in such a way that $b_2(z) = 0$, and

$$\varphi = -\exp\int \sigma_1\, dz.$$

Further, if the coefficients a_k are continuously differentiable once, then it follows from (5) that equation (14) is reducible by a change of the function $y = \varphi\tilde{y}$ to an equation with constant coefficients if and only if its projective invariants are constant; here one should put

$$\varphi = -\exp\int \sigma_1\, dt.$$

Finally, from Frenet' s formulas it follows that equation (14) is reducible by a change of variable and a linear change of the function to an equation with constant coefficients if and only if its invariants $\varkappa_k$ are constant; here it is required that the coefficients a_k be differentiable 3 times when $q < n-2$ and 4 times when $q = n-2$, although some of them may be differentiable a smaller number of times. The parameter is changed according to the formula

$$s = \int |v_q|^{1/q}\, dt,$$

and

$$\varphi = -\exp\int \Gamma(s)\, ds^*.$$

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* The results of the work were presented at the IV All-Union Mathematical Congress.

Note: Figure translations are in progress. See original paper for figures.

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