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MATHEMATICS

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Abstract

Full Text

MATHEMATICS

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ON THE GENERAL THEORY OF LINEAR FILTERS

(Presented by Academician V. I. Smirnov, 9 VI 1962)

1. By a **linear filter** Φ we shall mean a device that transforms the "input" $x(t)$ into the "output" $y(t)$ according to the formula

$$y(t) = \int_0^{\infty} x(t - \tau) dU(\tau). \quad (1)$$

The function $U(t)$, called the **transition function**, is the response of the filter to the unit input $x(t) = 1(t)$. If $U(t) \equiv 0$ for $t < k$ ($k \geq 0$), then k will be called the **delay time** of the filter Φ . The filter Φ is said to be **stable** if

$$\int_0^{\infty} |dU(\tau)| < \infty,$$

i.e., if $U(t)$ has bounded variation on the whole axis. The energy characteristic of stable filters is given by

Theorem 1. *In order that the filter Φ be stable, it is necessary and sufficient that every input $x(t)$, equal to zero for $t < 0$, for which*

$$\int_0^t |x(\tau)|^2 d\tau \leq Mt \quad (t > 0),$$

be transformed into an output $y(t)$ such that

$$\int_0^t |y(\tau)|^2 d\tau \leq M_1 t \quad (t > 0),$$

where M_1 depends only on M .

Let us note that for a stable filter the relation

$$\int_{-\infty}^{\infty} |U(t + \varepsilon) - U(t - \varepsilon)|^2 dt = O(\varepsilon) \quad (\varepsilon \rightarrow 0, \infty), \quad (2)$$

is valid; it shows that the energy of a rectangular impulse is preserved (in the sense of order of magnitude). However, this condition is not sufficient for stability, as is shown by the example of the transition function $U(t) = \frac{\sin t}{t} 1(t)$.

Definition. The filter Φ is called **weakly stable** if its transition function $U(t)$ is locally summable, if for every $\varepsilon > 0$ one has $U(t + \varepsilon) - U(t - \varepsilon) \in L^2$, and if (2) holds.

For weakly stable filters the convolution (1) must be understood in the generalized sense, but for inputs $x(t)$ of bounded variation and equal to zero for $t < t_0$, the definition remains the usual one.

If (2) holds only as $\varepsilon \rightarrow 0$ (or as $\varepsilon \rightarrow \infty$), then we shall call the filter Φ **weakly stable at zero** (respectively, **at infinity**). For example, for $U(t) = \ln^+ t$ or $(t - 1)^\alpha 1(t - 1)$ ($0 < \alpha < 1/2$) we have weak stability at zero, while if $U(t)$ is finite and belongs to the class L^2 , then we have weak stability at infinity (although it may not hold at zero).

Introduce the **frequency response** of the filter

$$F(w) = \int_0^\infty e^{-2\pi i \tau w} dU(\tau), \quad w = \omega + iy. \quad (3)$$

If Φ is weakly stable, then $F(w)$ should be understood as the Fourier transform of the generalized function $U'(t)$ (see, for example, (1)). The quantity $|F(\omega)|$ is called the **amplitude-frequency response**. If it does not vanish, then we shall write $F(\omega) = |F(\omega)|e^{-i\varphi(\omega)}$ and call $\varphi(\omega)$ the **phase-frequency response**.

Definition. The filter Φ is called **minimum-phase** if the delay time $k = 0$ and the frequency response $F(w)$ is continuous* and nonzero in the closed lower half-plane $y \leq 0$.

2. Let us consider the following questions:

A. On the ω -axis an even nonnegative function $A(\omega)$ is given. Does there exist a filter Φ having $A(\omega)$ as its amplitude-frequency response?

B. A filter Φ with frequency response $F(\omega)$ is given. Does there exist a minimum-phase filter Φ_0 having the same amplitude-frequency response $|F(\omega)|$?

C. A filter Φ with frequency response $F(\omega)$ is given. Does there exist an inverse filter Φ^{-1} , i.e., a filter with frequency response $\frac{1}{F(\omega)}$?

In the article by Bode and Shannon (2), p. 120, and in Goldman's monograph (3), p. 261, there are some assertions concerning problems A, B, and C; however, a rigorous analysis of them is lacking.

If one attempts to solve these questions for the class of stable filters, then one encounters specific difficulties arising from the absence of an effective character-

ization of the class of Fourier–Stieltjes transforms (3). For an affirmative answer to question A, necessary conditions are the continuity of $A(\omega)$ and the Wiener–Paley condition:

$$\int_{-\infty}^{\infty} \frac{|\ln A(\omega)|}{1 + \omega^2} d\omega < \infty, \quad (4)$$

but they are, of course, not sufficient.

If $F(w)$ has a finite number of zeros in the lower half-plane, then it can be shown that problem B has a positive solution in the class of stable filters. If, however, there is an infinite set of zeros a_i in the lower half-plane, a positive answer to B can be obtained only under rather special restrictions (absolute continuity of $U(t)$ and sufficiently rapid growth of the sequence $|\operatorname{Im} a_i|$).

As for problem C, by means of the Wiener–Pitt theorem (see, for example, ⁽⁴⁾, p. 213) and the use of Nevanlinna’s formula for a half-plane ⁽⁵⁾, one can obtain the following theorem:

Theorem 2. *If the filter Φ is stable and minimum-phase and the conditions*

$$\inf_{-\infty < \omega < \infty} |F(\omega)| > 0,$$

$$\operatorname{var} U_s(t) < \inf_{-\infty < \omega < \infty} \left| \int_0^{\infty} e^{-2\pi i \omega \tau} dU_d(\tau) \right|,$$

are satisfied:

* This is a restriction only in the case of a weakly stable filter.

where $U_s(t)$ is the singular part of $U(t)$, and $U_d(t)$ is its jump function, then there exists a stable inverse filter.

Corollary. If $U_s(t) \equiv 0$ and the jump of $U(t)$ at zero exceeds the sum of the absolute values of all the other jumps of $U(t)$, then the stable and minimum-phase filter Φ has a stable inverse filter Φ^{-1} .

3. Considerably more complete results are obtained if problems A, B, and C are posed in the broader class of weakly stable filters. Relying on the results of the author’s note ⁶, one can prove the following theorems:

Theorem 3. In order that an even nonnegative function $A(\omega)$ be the amplitude-frequency characteristic of some weakly stable filter, it is necessary and sufficient that condition (4) and the condition

$$\sup_N \frac{1}{N} \int_0^N A^2(\omega) d\omega < \infty. \quad (5)$$

be satisfied.

Theorem 4. If the frequency characteristic of a weakly stable filter is continuous in the closed half-plane $y \leq 0$, then there exists a weakly stable minimum-phase filter with the same amplitude-frequency characteristic.

Theorem 5. In order that a weakly stable filter Φ have a weakly stable inverse filter, it is necessary and sufficient that it be minimum-phase and that the condition

$$\sup_{-\infty < N < \infty} \frac{1}{N} \int_0^N \frac{d\omega}{|F(\omega)|^2} < \infty. \quad (6)$$

be satisfied.

These theorems show that, with respect to problems A, B, and C, the class of weakly stable filters is more natural than the class of stable filters.

In conclusion, we note that entirely analogous theorems can be obtained for filters that are weakly stable at zero or at infinity. In the first case the supremum in formulas (5) and (6) should be taken under the condition $|N| > 1$, and in the second under the condition $|N| < 1$, but the following requirement should be added:

$$\frac{A(\omega)}{1 + |\omega|} \in L^2, \quad \text{respectively} \quad \frac{1}{|F(\omega)|(1 + |\omega|)} \in L^2.$$

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CITED LITERATURE

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- ³ S. Goldman, *Information Theory*, II, 1957.
- ⁴ I. M. Gel'fand, D. A. Raikov, G. E. Shilov, *Commutative Normed Rings*, Moscow, 1960.
- ⁵ R. Nevanlinna, *Acta Soc. Sci. Fenn.*, **50**, No. 12 (1925).
- ⁶ N. S. Landkof, *DAN*, **147**, No. 3 (1962).

Note: Figure translations are in progress. See original paper for figures.

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