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Abstract

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ON THE PRINCIPAL DIRECTIONS OF AN m-DIMENSIONAL SURFACE

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The principal direction of a multidimensional surface is defined in ([1]) as a direction in which the vector of normal curvature of the surface assumes an extremal length different from zero. Although this concept, which is a direct generalization of the concept of a principal direction in the classical theory of surfaces, was introduced by I. A. Schouten as early as 1924, we nevertheless know very little about its role and significance for the geometry of a multidimensional surface.

In the present paper we derive a certain property of the principal direction connected with an infinitesimal displacement of the tangent plane to the surface V_m. The discussion is carried out in Euclidean space R_n. In the case V_2 and R_4, the infinitesimal displacement of the tangent plane was studied by Van Yung-chow ([2]).

1. Consider on $V_m \subset R_n$ an arbitrary point M and some neighborhood U of it. Let a curve passing through the point M be given in U; denote it by $M(s)$, where s is the arc length, and the point M corresponds to the value $s = 0$. Suppose, furthermore, that on this curve $M(s)$ a moving semiorthonormal frame $\{M(s), \mathbf{e}_a(s), \mathbf{e}_\alpha(s)\}$ is defined $(a, b, \ldots = 1, \ldots, m;\ \alpha = m+1, \ldots, n)$. The tangent plane to V_m at the point $M(s)$ is determined by the vectors $\mathbf{e}_a(s)$; denote it by $R_m(s)$. For simplicity, instead of $\mathbf{e}_a(0)$, $\mathbf{e}_\alpha(0)$, $R_m(0)$ we shall write, respectively, $\mathbf{e}_a$, $\mathbf{e}_\alpha$, R_m.

Choose in the planes R_m and $R_m(s)$ one freely varying unit vector each ($\equiv$ direction)

$$\mathbf{x} = x^a \mathbf{e}_a \in R_m, \qquad \tilde{\mathbf{x}} = \tilde{x}^a \mathbf{e}_a(s) \in R_m(s)$$

and denote by α the angle between them; then $\cos\alpha = (\mathbf{x}, \tilde{\mathbf{x}})$. For a given $s \neq 0$, on the planes R_m and $R_m(s)$ there are determined systems of orthogonal directions (not necessarily one-dimensional) $X(s)$ and $\widetilde{X}(s)$, respectively, between which the cosine (and hence also the square) of the angle α assumes stationary values ([3]). As the point $M(s)$ approaches the point M, the systems $X(s)$ and $\widetilde{X}(s)$ determine in R_m a certain limiting system

$$X = \lim_{s \to 0} X(s) = \lim_{s \to 0} \widetilde{X}(s).$$

Theorem 1. *The limiting system X in the plane R_m depends only on the point M and on the direction of the curve $M(s)$ at this point.*

Proof. The systems $X(s)$ and $\widetilde{X}(s)$ are orthogonal projections of one another in the plane $R_m(s)$ and R_m (^{3}). Consequently, the angles α having stationary square are realized between vectors of the form $\mathbf{x}$ and $\mathbf{x}^*(s)$, where $\mathbf{x}^*(s)$ is a vector in the plane $R_m(s)$ orthogonally projecting onto the plane R_m as the vector $\mathbf{x}$.

Since for sufficiently small values of s the function $\mathbf{e}_a(s)$ can be expanded in a series

$$\mathbf{e}_a(s) = \mathbf{e}_a + d\mathbf{e}_a + ... \, \mathbf{e}_a + \omega_a^b \mathbf{e}_b + \omega_a^\alpha \mathbf{e}_b \mathbf{e}_a + ...$$

and the linear relations between the vectors under projection are prese-

are, then, obviously, for the vector $\mathbf{x} = x^a \mathbf{e}_a$ the corresponding vector $\mathbf{x}^*$ has the form

$$\mathbf{x}^* = x^a \left(^\alpha a \mathbf{e}_\alpha + \omega_a^\alpha \mathbf{e}_\alpha + \mathbf{o}_a(s) \right), \tag{1}$$

where $\lim\limits_{s \to 0} \dfrac{1}{s} \mathbf{o}(s) = 0$.

Since for the surface the relations

$$\omega_a^\alpha = b_{ab}^\alpha \omega^b, \qquad b_{ab}^\alpha = b_{ba}^\alpha, \tag{2}$$

hold, then, using the notation $\mathbf{b}_{ab} = b_a^\alpha \mathbf{e}_\alpha$, the angle α between the vectors $\mathbf{x}$ and $\mathbf{x}^*$ can be expressed by the formula

$$\alpha = \operatorname{arc\,tg} |\mathbf{x}^* - \mathbf{x}| = \operatorname{arc\,tg} |\mathbf{b}_{ab} x^{a y^{bs}} + \mathbf{o}(s)|; \tag{3}$$

where y^b are the components of the unit vector $\mathbf{y} = y^a \mathbf{e}_a$ tangent to the curve $M(s)$ at the point M.

Now it is not difficult to show that the desired system X is determined by the solutions of the system

$$x^a \left(\mathbf{b}_{ac} \mathbf{b}_{bd} y_{ab}^{c y^{d - kg}} \right) = 0, \tag{4}$$

where k is a root of the equation

$$\operatorname{Det}|\mathbf{b}_{ac}\mathbf{b}_{bd}y_{ab}^{cy^{d-kg}}| = 0. \tag{5}$$

System (4) shows that X depends only on the point M and on the direction $\mathbf{y}$. The theorem is proved.

The directions forming the system X will be called **principal directions relative to the direction $\mathbf{y}$**. To each l-fold root of equation (5) there corresponds an l-dimensional principal direction relative to $\mathbf{y}$.

2. The vector $\vec{\xi} = \vec{\xi}(\mathbf{x}, \mathbf{y}) = \mathbf{b}_{ab}x^{ay^b}$ will be called the **vector of curvature of the surface V_m at the point M in the two directions x and y**. From (1) and (2) it follows that

$$\vec{\xi}(\mathbf{x}, \mathbf{y}) = \lim \frac{\mathbf{x}^* - \mathbf{x}}{s}.$$

This gives a geometric meaning to the vector $\vec{\xi}$. Obviously,

$$\vec{\xi}(\mathbf{x}, \mathbf{y}) = \vec{\xi}(\mathbf{x}, \mathbf{y}).$$

For fixed $\mathbf{y}$, the vector $\vec{\xi}(\mathbf{x}, \mathbf{y})$, as $\mathbf{x}$ varies, takes different values. The endpoints of such vectors $\vec{\xi}(\mathbf{x}, \mathbf{y})$, drawn from the point M, fill in space a certain geometric locus, which we shall call the **indicatrix of curvature of the surface V_m in the direction $\mathbf{y}$**.

The curvature indicatrix for $\mathbf{y}$ is either an $(m-1)$-dimensional ellipsoid, or else a degenerate figure obtained from such an ellipsoid if the lengths of some of its axes are made to tend to zero. To this figure we shall extend, by means of a limiting passage, all the concepts associated with an ellipsoid.

We omit the simple proofs of the following theorems.

Theorem 2. *The curvature indicatrix in the direction $\mathbf{y}$ has as many axes with lengths different from zero as there are linearly independent vectors among the vectors $\mathbf{b}_{ab}y^b$.*

Theorem 3. *The vectors $\vec{\xi}(\mathbf{x}, \mathbf{y})$ and $\mathbf{x}(\mathbf{x}', \mathbf{y})$ are conjugate radius-vectors of the curvature indicatrix in the direction $\mathbf{y}$ if and only if the directions $\mathbf{x}$ and $\mathbf{x}'$ are orthogonal.*

Theorem 4. *The direction $\mathbf{x}$ is a principal direction relative to $\mathbf{y}$ if and only if $\vec{\xi}(\mathbf{x}, \mathbf{y})$ is a principal radius-vector of the curvature indicatrix in the direction $\mathbf{y}$.*

Theorem 5. The roots of equation (5) are equal to the squares of the principal radius-vectors of the indicatrix of curvature in the direction $\mathbf{y}$.

For the proof of Theorem 5 it suffices to choose the frame so that its vectors are solutions of system (4).

Theorem 6. If the directions $\mathbf{x}$ and $\mathbf{y}$ on V_m are conjugate, i.e. if $b_{ab}x^a y^b = 0$ ([1]), then each of them is principal relative to the other.

The directions conjugate to the direction $\mathbf{y}$ correspond, obviously, to the roots of equation (5) that are equal to zero.

3. The vector $\vec{\xi}(\mathbf{x}, \mathbf{x})$ is called the **vector of normal curvature of the surface V_m in the direction $\mathbf{x}$**; the locus of the endpoints of these vectors, laid off from the point M, is called the **indicatrix of normal curvature of the surface V_m at the point M** ([1]).

Theorem 7. The indicatrix of normal curvature of the surface V_m at the point M is enveloped by the family of indicatrices of curvature corresponding to all possible tangent directions to the surface V_m at the point M.

The theorem follows from the relation

$$\frac{\partial}{\partial y^a}\,\vec{\xi}(\mathbf{y}, \mathbf{y}) = 2\,\frac{\partial}{\partial x^a}\,\vec{\xi}(\mathbf{x}, \mathbf{y})\Big|_{x=y}. \tag{6}$$

Theorem 8. A tangent direction to the surface is principal relative to itself if and only if the scalar square of the vector of normal curvature of the surface in this direction has a stationary value.

This theorem is a consequence of Theorems 4 and 7.

Remark 1. If in the tangent direction $\mathbf{x}$ the indicatrix of curvature is non-degenerate, then the requirement that the square of the vector $\vec{\xi}(\mathbf{x}, \mathbf{x})$ of normal curvature be stationary is equivalent to the condition that this vector $\vec{\xi}(\mathbf{x}, \mathbf{x})$ be orthogonal to the indicatrix of normal curvature.

4. A direction in which the vector of normal curvature of the surface has stationary length will be called a **principal direction of the surface**.

Remark 2. By requiring stationarity not of the scalar square (see 8), but of the length of the vector of normal curvature, we exclude from the number of directions that are principal relative to themselves those asymptotic directions in which the surface is tangentially non-degenerate ([4]). Thus, in the case R_3, our definition coincides completely with the classical definition of a principal direction.

There are cases in which the vector of normal curvature has stationary, but not extremal, length. A corresponding example is furnished by the vector of normal curvature $\vec{\xi}(\mathbf{x}, \mathbf{x})$ of a surface V_2 in R_4 ([5]), if the point M of the surface V_2 is an ordinary point of the evolute of the indicatrix (= ellipse) of normal curvature at this point, and the vector $\vec{\xi}(\mathbf{x}, \mathbf{x})$ is tangent to the evolute at the point M and orthogonal to the indicatrix.

Therefore the definition of a principal direction given by us is somewhat broader than that given in ([1]).

Theorem 9. Let there be given on the surface V_m a point M, the plane R_m tangent to V_m at M, a direction $\mathbf{y}$ in R_m, and a point M', obtained from the point M after an infinitesimal displacement of it along V_m in the direction $\mathbf{y}$. Let $\mathbf{x}$ be a freely varying direction in the plane R_m. The direction $\mathbf{y}$ is principal for the surface V_m if and only if the angle α between $\mathbf{x}$ and its orthogonal projection into the plane R'_m, tangent to V_m at M', assumes a stationary value for $\mathbf{x} = \mathbf{y}$. If the square of the angle α assumes a stationary value for $\mathbf{x} = \mathbf{y}$, then the direction $\mathbf{y}$ is principal relative to itself.

The proof of this theorem is based on relations (3) and (6). In the space R_3 this characteristic property of the principal direction is expressed by Rodrigues' formula.

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Note: Figure translations are in progress. See original paper for figures.