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Abstract

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EXACT SOLUTION OF THE FREEZING PROBLEM WITH AN ARBITRARY CHANGE OF TEMPERATURE AT A FIXED BOUNDARY

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The freezing problem ([1]) reduces to the solution of the system of equations

$$\frac{\partial u_i}{\partial t} = \frac{a_i t_0}{x_0^2} \frac{\partial^2 u_i}{\partial x^2} \quad (i = 1,\ 0 < x < \xi; \quad i = 2,\ \xi < x < \infty) \tag{1}$$

with the additional conditions

$$u_1(0, t) = f(t) < 0, \quad u_2(x, 0) = g(x) > 0, \quad \xi(0) = 0 \tag{2}$$

and the conditions at the moving boundary of the phase transition $\xi(t)$:

$$u_1\big|_\xi = u_2\big|_\xi = 0, \quad \left.\frac{\partial u_1}{\partial x}\right|_\xi - \frac{k_2}{k_1}\left.\frac{\partial u_2}{\partial x}\right|_\xi = \frac{\lambda \rho x_0^2}{k_1 t_0}\frac{d\xi}{dt}. \tag{3}$$

Here x, t are dimensionless coordinate and time; $u_i(x, t)$ is the temperature; $k_i,\ a_i$ are the coefficients of thermal conductivity and temperature conductivity; λ is the latent heat of phase transition; ρ is the density; the indices $1, 2$ refer respectively to the solid and liquid phases; x_0 and t_0 are arbitrarily chosen scales for measuring length and time.

Let us first consider the case when $u_2 \equiv 0$. We make the change of variables

$$t = \tau, \quad x = y\xi(\tau). \tag{4}$$

In the new variables equation (1) has the form (we omit the index $i = 1$ and put $a_1 t_0 = x_0^2$)

$$\xi^2(\tau)\frac{\partial u}{\partial \tau} - \xi(\tau)\xi'(\tau)y\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2}. \tag{5}$$

We apply to equation (5) the two-sided Laplace transform ([2]) with respect to the variable y, regarding τ as a parameter. After passing to the transforms we obtain

$$\xi^2(\tau)\frac{\partial v}{\partial \tau} + \xi(\tau)\xi'(\tau)p\frac{\partial v}{\partial p} = p^2 v. \tag{6}$$

The general solution of equation (6), under the assumption that

$$\xi(0)\xi'(0) \neq 0, \tag{7}$$

is given by the expression

$$v(p;\tau) = e^{p^2\tau/\xi^2(\tau)} F\left(\frac{p}{\xi}\right), \tag{8}$$

where $F(p/\xi)$ is an arbitrary function.*

We represent the function F in the form

$$F(p/\xi) = F_1(p/\xi) + F_2(p/\xi). \tag{9}$$

* The special solution of equation (6), when condition (7) is not satisfied ([3]), is obtained from the general one by a limiting transition and corresponds to the case of compatible initial and boundary conditions.

and we shall assume that F_1 and F_2 are images of arbitrary one-sided functions φ_1 and φ_2

$$F_1(p/\xi) = \varphi_1(\xi y)\,U(\xi y), \qquad F_2(p/\xi) = \varphi_2(\xi y)\,U(-\xi y). \tag{10}$$

Applying the multiplication rule for images ([2]), we find

$$u(y;\tau) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \exp\left[-\frac{\xi^2(y-\eta)^2}{4\tau}\right]\{\varphi_1(\xi\eta)U(\xi\eta) + \varphi_2(\xi\eta)U(-\xi\eta)\}\frac{\xi\,d\eta}{2\sqrt{\tau}}. \tag{11}$$

Thus the temperature field is determined by the Poisson integral, while the arbitrary functions $\varphi_1(x)U(x)$, $\varphi_2(x)U(-x)$ ($U(x)$ is the unit function) give the initial temperature distribution for $x > 0$ and $x < 0$, respectively.

To determine the unknown functions φ_1, φ_2 and ξ, we shall use conditions (2)–(3). Setting $\vartheta = 2\sqrt{\tau}$ in (11), we obtain

$$u(0,\vartheta) = f(\vartheta) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-\alpha^2} \varphi_1(\alpha\vartheta)\, d\alpha + \frac{1}{\sqrt{\pi}} \int_{-\infty}^0 e^{-\alpha^2} \varphi_2(\alpha\vartheta)\, d\alpha; \tag{12}$$

$$0 = \int_{-\xi/\vartheta}^\infty e^{-\beta^2} \varphi_1(\xi + \vartheta\beta)\, d\beta + \int_{-\infty}^{-\xi/\vartheta} e^{-\beta^2} \varphi_2(\xi + \vartheta\beta)\, d\beta; \tag{13}$$

$$\frac{\sqrt{\pi}\lambda\rho a}{k} \frac{d\xi}{d\vartheta} = \int_{-\xi/\vartheta}^\infty e^{-\beta^2} \varphi_1(\xi + \vartheta\beta)\, \beta\, d\beta + \int_{-\infty}^{-\xi/\vartheta} e^{-\beta^2} \varphi_2(\xi + \vartheta\beta)\, \beta\, d\beta. \tag{14}$$

Regarding the prescribed function $f(\vartheta)$ as an analytic function of ϑ,

$$f(\vartheta) = \sum_{k=0}^\infty f_k \vartheta^k, \tag{15}$$

we shall seek the solution of the problem in the form

$$\varphi_i(\omega) \sum_{k=0}^\infty \varphi_{ik} \omega^k \quad (i = 1, 2), \qquad \xi(\vartheta) = \sum_{k=1}^\infty \xi_k \vartheta^k. \tag{16}$$

Using asymptotic estimates of the corresponding integrals, it can be shown that the power series for $\varphi_i(\omega)$ have infinite radii of convergence. This entails uniform convergence of the indicated series and their boundedness for all ω, which, in turn, leads to uniform convergence in ϑ of the integrals entering (12)–(14), and makes it possible to differentiate these integrals with respect to the parameter ϑ any number of times. From the boundedness of the right-hand side of (14) follows the existence and boundedness of $d\xi/d\vartheta$. Differentiating both sides of (14) with respect to ϑ, we obtain that $d^2\xi/d\vartheta^2$ is also bounded. Continuing to differentiate both sides of (14) with respect to ϑ, we find that $\xi(\vartheta)$ is an analytic function of ϑ.

Thus, the series (16) give an exact solution of the problem under consideration. The coefficients φ_{in}, ξ_{n+1} for $n \geqslant 0$ are determined by a system of equations whose right-hand sides depend on the known coefficients f_k and φ_{ik}, ξ_{k+1} with index $k < n$:

$$[P_{1n} + (-1)^n P_{2n}]\varphi_{1n} + (h_1 - h_2)\xi_{n+1} =$$

$$= (-1)^n \frac{2f_n P_{2n}}{P_n} - \frac{1}{n!}(A_{1n} - A_{2n} + B_{1n} - B_{2n}),$$

$$[Q_{1n} + (-1)^n Q_{2n}]\varphi_{1n} - [(n+1)H + (h_1 - h_2)\xi_1]\xi_{n+1} =$$

$$= (-1)^n \frac{2f_n Q_{2n}}{\Phi_n} - \frac{1}{n!}(C_{1n} - C_{2n} - D_{1n} + D_{2n}), \tag{17}$$

$$\varphi_{2n} = (-1)^n \left(\frac{2f_n}{\Phi_n} - \varphi_{1n}\right) \qquad (H = \sqrt{\pi}\lambda\rho a/k).$$

Here

$$A_{in} = \int_{-\xi_1}^{\infty} e^{-\beta^2} R_{in}\, d\beta, \qquad B_{in} = \sum_{k=0}^{n-1} (S_{ik})_0^{(n-k-1)} - n!\, h_i \xi_{n+1},$$
$$C_{in} = \int_{-\xi_1}^{\infty} e^{-\beta^2} R_{in}\, \beta\, d\beta, \tag{18}$$

$$D_{in} = \sum_{k=0}^{n-1} (\gamma S_{ik})_0^{(n-k-1)} - n!\, h_i \xi_1 \xi_{n+1}, \qquad \Phi_k = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-\alpha^2} \alpha^k\, d\alpha, \qquad \gamma = \frac{\xi}{\vartheta};$$
$$P_{in} = \int_{-\xi_1}^{\infty} e^{-\beta^2} (\xi_1 + \beta)^n\, d\beta, \qquad Q_{in} = \int_{-\xi_1}^{\infty} e^{-\beta^2} (\xi_1 + \beta)^n \beta\, d\beta, \tag{19}$$

$$R_{in} = \left(\frac{d^n \varphi_i}{d\vartheta^n}\right)_0 - n!\, \varphi_{in} (\xi_1 + \beta)^n, \qquad S_{ik} = \gamma' e^{-\gamma^2} \frac{d^k \varphi_i}{d\vartheta^k}\bigg|_{\beta = -\gamma},$$
$$h_i = e^{-\xi_1^2} \varphi_{i0}, \qquad \nu = (-1)^{i-1}\ (i = 1, 2), \qquad B_{i0} = -h_i \xi_1 = \frac{1}{\xi_1} D_{i0}.$$

Hence, for $n = 0$ we obtain the known solution corresponding to the case $f(\vartheta) = f_0 = \text{const}$ ([1]):

$$\varphi_{10} = f_0 \left(1 - \frac{1}{\operatorname{erf} \xi_1}\right), \qquad \varphi_{20} = f_0 \left(1 + \frac{1}{\operatorname{erf} \xi_1}\right), \qquad -\lambda\rho\sqrt{\pi}\xi_1 = \frac{k f_0}{a} \frac{e^{-\xi_1^2}}{\operatorname{erf} \xi_1}. \tag{20}$$

The coefficients of the series (16) representing the solution are given by formulas (20), and for $n > 0$ by the solutions of the system of linear algebraic equations

(17). The uniqueness of the solution constructed is obvious. For $n = 1$ these coefficients have the form*

$$\xi_2 = -\frac{\sqrt{\pi}e^{-\xi_1^2} f_1}{2H\xi_1(3+2\xi_1^2)}, \qquad \varphi_{11}\sqrt{\pi}f_1 \operatorname{erfc}\xi_1 + \frac{4H}{\sqrt{\pi}}(1+\xi_1^2)\xi_2, \qquad \varphi_{21} = \varphi_{11} - 2\sqrt{\pi}f_1. \tag{21}$$

Suppose that $f(\vartheta)$, continued to the plane of a complex variable, is an entire function of exponential type. Then $\varphi_i(\omega)$ will also be entire functions of exponential type and, consequently, F_i will be functions regular at infinity. Their Laurent expansions in a neighborhood of the infinitely distant point have the form

$$F_i(\xi/p) = F_{i0} + F_{i1}\xi/p + F_{i2}(\xi/p)^2 + \cdots \tag{22}$$

Using the expansions (22) and applying the integration rule (^{2}) for $i = 1$ when $\operatorname{Re} p > 0$, and for $i = 2$ when $\operatorname{Re} p < 0$, we find

$$u(y,\vartheta) = \frac{1}{2}\sum_{k=0}^{\infty} \vartheta^k \{F_{1k} i^k \operatorname{erfc}(-\gamma y) + (-1)^{k+1} F_{2k} i^k \operatorname{erfc}(\gamma y)\}. \tag{23}$$

where

$$i^n \operatorname{erfc} x = \int_x^{\infty} i^{\,n-1} \operatorname{erfc}\alpha \, d\alpha, \qquad i^0 \operatorname{erfc} x = \operatorname{erfc} x = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-\alpha^2}\, d\alpha, \tag{24}$$

$$2ni^n \operatorname{erfc} x = i^{\,n-2} \operatorname{erfc} x - 2x i^{\,n-1} \operatorname{erfc} x.$$

The solution in the form (23) is convenient for small values of the parameter $\gamma = \xi/\vartheta$. By virtue of conditions (2), (3), $u(0,\tau) = f(\tau)$, $u(1,\tau) = 0$. Consequently,

$$-f(\tau) = \frac{1}{2}\sum_{k=0}^{\infty} \vartheta^k \{F_{1k}[i^k \operatorname{erfc}(-\gamma) - i^k \operatorname{erfc} 0] + (-1)^{k+1} F_{2k}[i^k \operatorname{erfc}\gamma - i^k \operatorname{erfc} 0]\}. \tag{25}$$

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* The following coefficients are not given because of lack of space. From this solution, by a limiting passage, exact solutions with compatible initial and boundary

conditions are obtained. In particular, when $f_0 \to 0$ and $\xi_k = 0$ $(k \geq 3)$, one obtains a solution with a uniformly moving wave $\xi = \beta t$.

Differentiating (24) with respect to y, we obtain, for $y = 1$,

$$\left.\frac{\partial u}{\partial y}\right|_{y=1} = \frac{1}{2}\gamma \left\{ \frac{2}{\sqrt{\pi}} e^{-\gamma^2} (F_{10} + F_{20}) + \sum_{k=1}^{\infty} \vartheta^k \left[F_{1k} i^{k-1} \operatorname{erfc}(-\gamma) + (-1)^k F_{2k} i^{k-1} \operatorname{erfc} \gamma \right] \right\}. \tag{26}$$

From (25) and (26) it follows that, to within terms of order $\gamma^2\vartheta$ and higher,

$$\left.\frac{\partial u}{\partial y}\right|_{y=1} = -f(\tau). \tag{27}$$

Substituting (27) into the last of conditions (3) and integrating with respect to τ, we obtain a convenient approximate formula expressing the law of motion of the phase-transition surface for the case when terms of order $\gamma^2\vartheta$ may be neglected:

$$\xi(\tau) = \left[-\frac{2k}{\lambda \rho a} \int_0^{\tau} f(\tau)\, d\tau \right]^{1/2}. \tag{28}$$

In the general case, when

$$u_2(x, 0) = g(x) = \sum_{k=0}^{\infty} \frac{1}{k!} g_k x^k \qquad (g(x) \neq 0,\ x > 0), \tag{29}$$

the exact solution in series is constructed analogously. An approximate solution for small values of the parameter γ can be obtained by using conditions (29), (3) and a solution in a form analogous to (23):

$$u_2(y, \vartheta) = \frac{1}{2} \sum_{k=0}^{\infty} \vartheta^k \left(\frac{a_2}{a_1} \right)^{k/2} \left[H_{1k} i^k \operatorname{erfc} \left(-\gamma \sqrt{\frac{a_1}{a_2}}\, y \right) + (-1)^{k+1} H_{2k} i^k \operatorname{erfc} \left(\gamma \sqrt{\frac{a_1}{a_2}}\, y \right) \right]. \tag{30}$$

Considering (29) as a limiting condition as $y \to \infty$, $\tau \to 0$, we obtain that $H_{1k} = g_k$.

Taking into account the second of conditions (3), we find, to within terms of order $\gamma^2\vartheta$ and higher,

$$\left.\frac{\partial u_2}{\partial y}\right|_{y=1} = \frac{1}{2} \sum_{k=0}^{\infty} \vartheta^k \left(\frac{a_2}{a_1} \right)^{k/2} (-1)^k \frac{H_{2k} - (-1)^k g_k}{2^k \Pi(k/2)}. \tag{31}$$

Substituting (27) and (31) into the last of conditions (3) and integrating with respect to τ, we obtain

$$\frac{\xi^2}{4\tau} = \gamma^2 = -\frac{k_1}{2\lambda\rho a_1}\left\{\frac{1}{\tau}\int_0^\tau f(\tau)\,d\tau + \frac{k_2}{k_1}\sum_{n=0}^{\infty}(-1)^n(H_{2n} - (-1)^n g_n)\left(\frac{a_2}{a_1}\right)^{n/2}\frac{\tau^{n/2}}{(n+2)\Pi(n/2)}\right\}. \quad (32)$$

The approximate values $\gamma_0^{(n)}$ and H_{2n} are determined from relation (32) and the identity $u_2(1,\vartheta) \equiv 0$.

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Note: Figure translations are in progress. See original paper for figures.