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Abstract

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MATHEMATICS

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ASYMPTOTIC COMPUTATION OF PERIODIC SOLUTIONS CLOSE TO DISCONTINUOUS ONES FOR SYSTEMS OF SECOND-ORDER DIFFERENTIAL EQUATIONS

(Presented by Academician L. S. Pontryagin on 12 II 1962)

It is known (7) that, under fairly general assumptions on the right-hand sides, the system of second-order equations

$$\varepsilon \dot{x} = f(x, y), \qquad \dot{y} = g(x, y) \tag{1}$$

may have a periodic solution Z_ε, "close" to a discontinuous periodic solution Z_0. The most typical picture in the phase plane is presented in Fig. 1, where the solution Z_0 is indicated by a heavy line, and Z_ε by a dashed line.

The solution Z_ε consists of portions of two types: portions of "slow motions," lying near the arcs AB and CD of the curve $f(x, y) = 0$ and traversed by the representing point in a finite time, and portions of "fast motions," lying near the horizontal segments BC and DA and traversed by the representing point in a time small together with ε.

Fig. 1

Fig. 1

Many works have been devoted to the asymptotic computation of solutions of system (1) for small $\varepsilon > 0$ and, in particular, solutions of the type Z_ε (see, for example, (1, 2, 4–6)). The most general results were obtained in (3). There, with arbitrary accuracy, asymptotic expansions were found for the solution on portions of "slow" and "fast" motions and in neighborhoods of the points of loss of stability (points B and D in Fig. 1) and of the points of falling (points A and C). Then these expansions were "matched" with accuracy up to $O(\varepsilon^{7/6})$ at the junctions of the portions. From the expansion found for the solution Z_ε, an asymptotic formula is obtained for the period T_ε of the relaxation oscillation Z_ε:

$$T_\varepsilon = T_0 + Q_1\varepsilon^{2/3} + Q_2\varepsilon\ln\varepsilon + Q_3\varepsilon + O(\varepsilon^{7/6}). \tag{2}$$

Here T_0 is the period of the discontinuous solution Z_0, and Q_1, Q_2, Q_3 are numerical coefficients depending on the values of the right-hand sides of system (1) and their derivatives on the arcs AB and CD of the curve $f(x,y)=0$.

In the present work a complete asymptotic expansion of the period of the relaxation oscillation Z_ε is obtained, i.e., the period of the cycle Z_ε is computed with any prescribed degree of accuracy $o(\varepsilon^N)$.

It turns out that*

$$T_\varepsilon = T_0 + \sum_{n=2}^{3N} \varepsilon^{n/3} \sum_{\nu=0}^{[n/3]-\chi(n)} Q_{n,\nu}\ln^\nu\varepsilon + o(\varepsilon^N). \tag{3}$$

In this formula T_0 is the same as in (2); $\chi(n)=0$, if $n\equiv 0 \pmod 3$ or $n\equiv 2 \pmod 3$ and $\chi(n)=1$, if $n\equiv 1 \pmod 3$; $Q_{n,\nu}$ are numerical coefficients whose values are determined by the values of the right-hand sides of system (1) and of several of their derivatives on the arcs AB and CD of the curve $f(x,y)=0$.

Concrete explicit expressions have been obtained for the coefficients in the formula

$$T_\varepsilon = T_0 + Q_1\varepsilon^{2/3} + Q_2\varepsilon\ln\varepsilon + Q_3\varepsilon + Q_4\varepsilon^{4/3} + Q_5\varepsilon^{5/3}\ln\varepsilon + Q_6\varepsilon^{5/3} +$$
$$+ Q_7\varepsilon^2\ln^2\varepsilon + Q_8\varepsilon^2\ln\varepsilon + O(\varepsilon^2).$$

Effective expressions can also be obtained for the subsequent coefficients, but the computations necessary for this are too cumbersome.

If, instead of system (1), one considers the more general system

$$\varepsilon\dot{x} = f(x,y,\varepsilon), \qquad \dot{y} = g(x,y,\varepsilon), \tag{4}$$

then, although the formulas for the solutions change, the structure of the asymptotic expansion (3) is preserved, and the coefficients $Q_{2,0}$ and $Q_{3,1}$ do not even change.

The derivation of formula (3) is based on refining the asymptotic expansions (obtained in [3]) of the solution Z_ε on various intervals and on "matching" these expansions with an arbitrary degree of accuracy.

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* The symbol $[a]$ denotes the integer part of the number a.

The greatest difficulty is presented by the computation of the time $T_{-p,p}$ spent by the representative point in traversing part of the trajectory in a finite (independent of ε) neighborhood of the jump point. Introducing (see [3]) in this neighborhood special local coordinates ξ, η, we can determine the time

$$T_{-p,p} = \int_{-p}^{p} \frac{d\eta}{d\xi}\,\delta(\xi,\eta)\,d\xi, \tag{5}$$

where $\eta = \eta(\xi)$ is the equation of a piece of the cycle in local coordinates, and $\delta(\xi,\eta)$ is a certain function expressible in terms of the right-hand sides of system (1). The asymptotic representation for the function $\eta = \eta(\xi)$ is obtained differently on different intervals: on the interval $[-p, -\sigma_1]$, $\sigma_1 = \varepsilon^{\lambda}$, the solution is asymptotically represented in the form

$$\eta(\xi) = \sum_{i=0}^{n} \varepsilon^{i}\eta_i(\xi) + O\big(\varepsilon^{\,n(1-3\lambda)+1-\lambda}\big)\,;$$

on the interval $[-\sigma_1, 0]$—in the form

$$\eta(\xi) = \mu^2 \sum_{i=0}^{k} \mu^{i} v_i\left(\frac{\xi}{\mu}\right) + O\big(\varepsilon^{1+k\lambda}\big)\,,$$

where $\mu^3 = \gamma\varepsilon$, γ is a certain constant; on the interval $[0, \sigma_2]$, $\sigma_2 = \varepsilon^{\nu}$,—in the form

$$\eta(\xi) = \mu^2 \sum_{i=0}^{k} \mu^{i} v_i\left(\frac{\xi}{\mu}\right) + O\big(\varepsilon^{1+k\nu}\big)\,;$$

on the interval $[\sigma_2, p]$—in the form

$$\eta(\xi) = \eta_{(l)}(\xi) + O\big(\varepsilon^{\,l(1-3\nu)+1-\nu}\big)\,.$$

For all the functions $\eta_i(\xi)$, $v_i(\xi/\mu)$, $\eta_{(l)}(\xi)$, effective algorithms of computation are indicated in [3].

It is necessary to choose the numbers n, λ, k, ν, l in such a way that the remainder terms have order $O(\varepsilon^{N})$ or higher. It turns out that such a choice of these numbers is possible, and at the same time at the points $-\sigma_1$ and σ_2 the pieces of the solutions can be matched. Taking into account the asymptotic expansions of the functions $v_i(u)$ for

for large negative values of u

$$v_i(u)^- \sim u^{i-1}\sum_{\alpha=0}^{\infty}\frac{a_{\alpha,i}}{u^{3\alpha}}$$

and the functions $\eta_i(\xi)$ as $\xi \to 0$

$$\eta_i(\xi) \sim \sum_{\alpha=0}^{\infty} b_{\alpha,i}\xi^{\alpha-3i+2},$$

the time $T_{-p,0}$ is computed by formula (5). Determination of $T_{0,p}$ is more complicated because of the peculiar asymptotics of $v_i(u)$ as $u \to +\infty$:

$$v_i(u)^+ \sim \sum_{\alpha=0}^{i}\sum_{\substack{\beta=1+3\alpha-i\\ \beta\neq 2+3\alpha-i}}^{\infty}\frac{A_{\alpha,\beta,i}\ln^{\alpha}u}{u^{\beta}} + \Phi_i(u), \tag{6}$$

where

$$\Phi_i(u) = \begin{cases} A_i \ln^{\rho} u, & \text{if } i = 3\rho, \\ A_i \ln^{\rho+1} u, & \text{if } i = 3\rho+1, \\ A_i \ln^{\rho} u, & \text{if } i = 3\rho+2. \end{cases}$$

However, on the basis of formulas (6) and (5) it too can be found.

The computation of the time of motion of the representative point on the remaining parts of the trajectory Z_ε is carried out on the basis of asymptotic representations obtained analogously to those indicated in paper (^{3}).

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REFERENCES CITED

[1] J. Haag, Ann. Sci. École Norm. Sup., **60**, 35 (1943); **61**, 65 (1944).
[2] A. A. Dorodnitsyn, Prikl. matem. i mekh., **11**, No. 3, 313 (1947).
[3] E. F. Mishchenko, Matem. sborn., **44**, No. 4, 457 (1958).
[4] L. S. Pontryagin, Izv. AN SSSR, ser. matem., **21**, 605 (1957).
[5] E. F. Mishchenko, Izv. AN SSSR, ser. matem., **21**, 627 (1957).
[6] A. N. Tikhonov, Matem. sborn., **31**, No. 3, 574 (1952).

[7] A. A. Andronov, A. A. Vitt, S. E. Khaikin, *Theory of Oscillations*, Moscow, 1959.

Note: Figure translations are in progress. See original paper for figures.

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