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Abstract

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ON DETERMINING THE POSITION OF A NONLINEAR CONTROLLED SYSTEM IN PHASE SPACE

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To determine the position of a linear controlled system in phase space in the case where not all phase coordinates are accessible to measurement and, moreover, there is no information about the position of the reference system relative to which the position of the controlled system is to be determined, an indirect method was developed in paper ([1]). As the initial data, the increments of one of the phase coordinates of the system accessible to measurement were used. In the present paper an analogous problem is considered for nonlinear controlled systems.

1. The equations of motion of a controlled system of continuous action may be represented in the following form:

$$\sum_{k=1}^{n} f_{jk}(D)y_k = x_j(t)+\psi_j(y_1,\dot{y}_1,\dots,y_1^{(m_1-1)},\dots,y_n,\dot{y}_n,\dots,y_n^{(m_n-1)},t) \quad (j=1,\dots,n). \tag{1.1}$$

Here y_k are generalized coordinates, $x_j(t)$ are prescribed external forces. By $f_{jk}(D)$ are denoted polynomials in D, whose coefficients are prescribed functions of time; $D = d/dt$ is the operator of differentiation with respect to time. The highest degree of D in the polynomials $f_{jk}(D)$ $(j = 1,\dots,n)$ for a given k is denoted by m_k, i.e. m_k $(k = 1,\dots,n)$ is the order of the highest derivative of y_k with respect to time occurring in the left-hand sides of equations (1.1).

The nonlinear functions ψ_j $(j = 1,\dots,n)$ are assumed to be continuous in all their arguments in some closed domain and to satisfy in this domain Lipschitz conditions with respect to the arguments

$$y_1,\dot{y}_1,\dots,y_1^{(m_1-1)},\dots,y_n,\dot{y}_n,\dots,y_n^{(m_n-1)}.$$

Equations (1.1) can be reduced ([2]) to the following form:

$$\dot{z}_j + \sum_{k=1}^{r} a_{jk}(t) z_k = X_j(t) + \Psi_j(z_1, \dots, z_r, t) \quad (j = 1, \dots, r). \tag{1.2}$$

Here

$$z_1 = y_1, \quad z_2 = \dot{y}_1, \dots, \quad z_{m_1} = y_1^{(m_1-1)}, \dots, \quad z_r = y_n^{(m_n-1)}; \tag{1.3}$$

$$r = m_1 + m_2 + \dots + m_n; \tag{1.4}$$

$$X_{\sigma_j}(t) = \sum_{k=1}^{n} \frac{B_{kj}(t)}{\Delta^*(t)} x_k(t),$$

$$\Psi_{\sigma_j}(z_1, \dots, z_r, t) = \sum_{k=1}^{n} \frac{B_{kj}(t)}{\Delta^*(t)} \psi_k(z_1, \dots, z_r, t) \quad (\sigma_j = \sigma_1, \dots, \sigma_n); \tag{1.5}$$

$$\sigma_1 = m_1, \quad \sigma_2 = m_1 + m_2, \dots, \quad \sigma_n = r, \tag{1.6}$$

and the functions $X_\mu(t)$, $\Psi_\mu(z_1, \dots, z_r, t)$, for which $\mu \neq \sigma_l$ $(l = 1, \dots, n)$, are identically equal to zero.

In expressions (1.5), $\Delta^*(t)$ denotes the determinant formed from the coefficients $b_{jk}(t)$ with which the quantities $y_k^{(m_k)}$ enter the left-hand sides of equations (1.1),

$$\Delta^*(t) = |b_{jk}(t)|, \tag{1.7}$$

and it is assumed that this determinant is not identically equal to zero. By B_{kj} are denoted the cofactors of the elements b_{kj} in the determinant (1.7).

The system of scalar differential equations (1.2) can be replaced by the matrix differential equation

$$\dot{z} + a(t) z = X(t) + \Psi(z_1, \dots, z_r, t), \tag{1.8}$$

where z, $a(t)$, $X(t)$, $\Psi(z_1, \dots, z_r, t)$ are the following matrices:

$$z = \|z_j\|, \qquad a(t) = \|a_{jk}(t)\|,$$

$$X(t) = \|X_j(t)\|, \qquad \Psi(z_1, \dots, z_r, t) = \|\Psi_j(z_1, \dots, z_r, t)\|. \tag{1.9}$$

By $z(t_0) = \|z_j(t_0)\|$ we denote the matrix of values of the desired functions at the initial time $t = t_0$.

Denoting by $\theta(t)$ the fundamental matrix for the matrix differential equation

$$\dot{z} + a(t)z = 0, \tag{1.10}$$

one can pass from the nonlinear matrix differential equation (1.8) to the nonlinear matrix integral equation

$$z(t) = N(t,t_0)z(t_0) + \int_{t_0}^{t} N(t,\tau)X(\tau)\,d\tau + \int_{t_0}^{t} N(t,\tau)\Psi(z_1(\tau),\dots,z_r(\tau),\tau)\,d\tau, \tag{1.11}$$

where

$$N(t,\tau) = \theta(t)\theta^{-1}(\tau) \tag{1.12}$$

is the weight function for the matrix differential equation (1.10). By $\theta^{-1}(t)$ in expression (1.12) is denoted the inverse matrix.

Since the functions $X_\mu(t)$, $\Psi_\mu(z_1,\dots,z_r,t)$, for which $\mu \neq \sigma_l$ $(l = 1,\dots,n)$, are identically equal to zero, the system of scalar integral equations equivalent to the matrix equation (1.11) takes the form

$$z_j(t) = \sum_{k=1}^{r} N_{jk}(t,t_0)z_k(t_0) + \sum_{l=1}^{n}\int_{t_0}^{t} W_{jl}(t,\tau)x_l(\tau)\,d\tau + \sum_{l=1}^{n}\int_{t_0}^{t} W_{jl}(t,\tau)\psi_l(z_1(\tau),\dots,z_r(\tau),\tau)\,d\tau \quad (j = 1,\dots,r), \tag{1.13}$$

where

$$W_{jl}(t,\tau) = \sum_{i=1}^{n} N_{j\sigma_i}(t,\tau)\frac{B_{li}(\tau)}{\Delta^*(\tau)} \quad (j = 1,\dots,r;\ l = 1,\dots,n). \tag{1.14}$$

To solve the system of integral equations (1.13), it is necessary to prescribe the initial values $z_k(t_0)$ $(k = 1,\dots,r)$ of the phase coordinates. We are concerned, however, the problem of interest will be one in which the initial values $z_k(t_0)$ $(k = 1,\dots,r)$ are unknown, while it is possible to measure only one of the phase coordinates z_s, and the position of the origin of this coordinate is unknown.

We shall assume that the law according to which the external forces $x_l(t)$ $(l = 1,\dots,n)$ vary is known.

To solve the problem posed, let us choose a new arbitrary origin and, fixing it, measure the deviations $S(t_1), S(t_2), \dots, S(t_{r+1})$ of the phase coordinate z_s relative to the new origin at certain instants of time $t_1, \dots, t_{r+1}$.

Since $S(t_i) = S^* + z_s(t_i)$ $(i = 1, \dots, r+1)$, where S^* is the deviation of the new origin relative to the original origin, then, denoting

$$S(t_{\mu+1}) - S(t_\mu) = L_\mu \qquad (\mu = 1, \dots, r), \tag{1.15}$$

we obtain the following relations, which do not contain the unknown quantity S^*, between the increments of the phase coordinate z_s and the measurement results L_μ:

$$z_s(t_{\mu+1}) - z_s(t_\mu) = L_\mu \qquad (\mu = 1, \dots, r). \tag{1.16}$$

With the aid of (1.13), the expressions (1.16) can be brought to the following form:

$$\sum_{k=1}^{r} c_{\mu k} z_k(t_0) = P_\mu - \sum_{l=1}^{n} \int_{t_0}^{t_{\mu+1}} W_{sl}(t_{\mu+1}, \tau)\, \psi_l(z_1(\tau), \dots, z_r(\tau), \tau)\, d\tau + \sum_{l=1}^{n} \int_{t_0}^{t_\mu} W_{sl}(t_\mu, \tau)\, \psi_l(z_1(\tau), \dots, z_r(\tau), \tau)\, d\tau \tag{1.17}$$

where

$$c_{\mu k} = N_{sk}(t_{\mu+1}, t_0) - N_{sk}(t_\mu, t_0) \qquad (\mu = 1, \dots, r;\ k = 1, \dots, r); \tag{1.18}$$

$$P_\mu = L_\mu - \sum_{l=1}^{n} \int_{t_0}^{t_{\mu+1}} W_{sl}(t_{\mu+1}, \tau)\, x_l(\tau)\, d\tau + \sum_{l=1}^{n} \int_{t_0}^{t_\mu} W_{sl}(t_\mu, \tau)\, x_l(\tau)\, d\tau \qquad (\mu = 1, \dots, r). \tag{1.19}$$

From equations (1.17) it follows that

$$z_i(t_0) = \frac{1}{\Lambda} \sum_{\mu=1}^{r} A_{\mu i} P_\mu - \frac{1}{\Lambda} \sum_{\mu=1}^{r} A_{\mu i} \left[\sum_{l=1}^{n} \int_{t_0}^{t_{\mu+1}} W_{sl}(t_{\mu+1}, \tau)\, \psi_l(z_1(\tau), \dots, z_r(\tau), \tau)\, d\tau - \sum_{l=1}^{n} \int_{t_0}^{t_\mu} W_{sl}(t_\mu, \tau)\, \psi_l(z_1(\tau) \tag{1.20}$$

where

$$\Lambda = \begin{vmatrix} c_{11} & c_{12} & \dots & c_{1r} \\ \vdots & \vdots & \ddots & \vdots \\ c_{r1} & c_{r2} & \dots & c_{rr} \end{vmatrix}, \tag{1.21}$$

and $A_{\mu i}$ $(\mu, i = 1, \dots, r)$ are the cofactors of the elements $c_{\mu i}$ in the determinant (1.21).

Substituting into (1.13) the expressions (1.20) found for $z_i(t_0)$, we arrive at a system of integral equations, not containing the unknown initial values $z_k(t_0)$, with respect to the unknown functions $z_\nu(t)$ $(\nu = 1, \dots, r)$:

$$
\begin{aligned}
z_j(t) = G_j(t) - \sum_{\mu=1}^{r} V_{j\mu}(t) \Bigg[&\sum_{l=1}^{n} \int_{t_0}^{t_{\mu+1}} W_{sl}(t_{\mu+1}, \tau)\, \psi_l(z_1(\tau), \dots, z_r(\tau), \tau)\, d\tau \\
&- \sum_{l=1}^{n} \int_{t_0}^{t_\mu} W_{sl}(t_\mu, \tau)\, \psi_l(z_1(\tau), \dots, z_r(\tau), \tau)\, d\tau \Bigg] \\
&+ \sum_{l=1}^{n} \int_{t_0}^{t} W_{jl}(t, \tau)\, \psi_l(z_1(\tau), \dots, z_r(\tau), \tau)\, d\tau \qquad (j = 1, \dots, r),
\end{aligned}
\tag{1.22}
$$

where

$$
G_j(t) = \frac{1}{\Lambda} \sum_{i=1}^{r} \sum_{\mu=1}^{r} N_{ji}(t, t_0) A_{\mu i} P_\mu + \sum_{l=1}^{n} \int_{t_0}^{t} W_{jl}(t, \tau) x_l(\tau)\, d\tau \quad (j = 1, \dots, r);
\tag{1.23}
$$

$$
V_{j\mu}(t) = \frac{1}{\Lambda} \sum_{i=1}^{r} N_{ji}(t, t_0) A_{\mu i} \quad (j, \mu = 1, \dots, r).
\tag{1.24}
$$

To solve the integral equations (1.22), it is necessary to use numerical methods ([3],[4]).

2. In the case where the nonlinear functions $\psi_l(z_1, \dots, z_r, t)$ $(l = 1, \dots, n)$ do not depend on certain phase coordinates of the system z_ρ, the number of integral equations forming the system (1.22) is reduced. Thus, for example, if only one phase coordinate z_k enters under the sign of the nonlinear functions ψ_l $(l = 1, \dots, n)$,

$$
\psi_l = \psi_l(z_k(t), t) \qquad (l = 1, \dots, n),
\tag{2.1}
$$

then, in accordance with (1.22), it will be necessary to solve the following nonlinear integral equation with respect to the unknown function $z_k(t)$:

$$z_k(t) = G_k(t) - \sum_{\mu=1}^{r} V_{k\mu}(t) \left[\sum_{l=1}^{n} \int_{t_0}^{t_{\mu+1}} W_{sl}(t_{\mu+1}, \tau)\, \psi_l(z_k(\tau), \tau)\, d\tau - \sum_{l=1}^{n} \int_{t_0}^{t_{\mu}} W_{sl}(t_{\mu}, \tau)\, \psi_l(z_k(\tau), \tau)\, d\tau \right] + \sum_{l=1}^{n} \int_{t_0}^{t} W_{kl}(t, \tau)\, \psi_l(z_k(\tau), \tau)\, d\tau. \quad (2.2)$$

The remaining phase coordinates z_ρ $(\rho = 1, \dots, k{-}1, k{+}1, \dots, r)$ will be expressed by quadratures:

$$z_\rho(t) = G_\rho(t) - \sum_{\mu=1}^{r} V_{\rho\mu}(t) \left[\sum_{l=1}^{n} \int_{t_0}^{t_{\mu+1}} W_{sl}(t_{\mu+1}, \tau)\, \psi_l(z_k(\tau), \tau)\, d\tau - \sum_{l=1}^{n} \int_{t_0}^{t_{\mu}} W_{sl}(t_{\mu}, \tau)\, \psi_l(z_k(\tau), \tau)\, d\tau \right] + \sum_{l=1}^{n} \int_{t_0}^{t} W_{\rho l}(t, \tau)\, \psi_l(z_k(\tau), \tau)\, d\tau. \quad (2.3)$$

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Note: Figure translations are in progress. See original paper for figures.