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Abstract

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ON THE EXTENSION OF FUNCTIONS

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In the theory of boundary-value problems for partial differential equations, an essential role is played by the question of the possibility of extending a function $u(x)$, $x = (x_1, \dots, x_n)$, given in some bounded domain Q, to the whole space while preserving its smoothness. A function $u^*(x)$, defined on the whole space R_n, will be called an **extension from the domain Q of the function** $u(x)$, if $u^*(x) = u(x)$ for $x \in Q$. When $u(x)$ belongs to the Sobolev class $W_p^{(l)}(Q)$ (^{1}) for some integer l, and the boundary Γ of the domain Q is sufficiently smooth, its extension $u^*(x)$ to R_n, belonging to the class $W_p^{(l)}(R_n)$, was constructed by V. M. Babich (^{2}) by the methods of Whitney and Hestenes (^{3}). It was also proved that there exists a constant C, independent of the function $u(x)$, such that

$$\|u^*\|_{W_p^{(l)}(R_n)} \le C \|u\|_{W_p^{(l)}(Q)} . \tag{1}$$

L. N. Slobodetskii (^{4}) investigated an analogous question on the extension of functions from the class $W^{(l_1,\dots,l_n)}_{(x_1,\dots,x_n),2}(Q)$, where the l_i are arbitrary positive numbers, to the whole space in the case when Q is a domain of special form. If all l_i, $i = 1, \dots, n$, are distinct, then the domain Q in (^{4}) is assumed to be a parallelepiped with faces parallel to the coordinate planes. If $\{l_i,\ i = 1, \dots, n\}$ are divided into k groups of numbers equal among themselves $l_1 = l_2 = \dots = l_{s_1}$, $l_{s_1+1} = \dots = l_{s_2}, \dots, l_{s_{k-1}+1} = \dots = l_n$, then the domain Q in (^{4}) is the intersection of cylinders $\Omega_1, \dots, \Omega_k$,

$$Q = \prod_{r=1}^{k} Q_r,$$

where Ω_r is a cylinder with generators parallel to the coordinate planes $x_{s_{r-1}+1} = x_{s_{r-1}+2} = \dots = x_{s_r} = 0$, and with a sufficiently smooth directrix surface lying in the plane $\{x_{s_{r-1}+1}, \dots, x_{s_r}\}$. In (^{5}), in solving a certain boundary-value problem for a parabolic equation of order $2p$, an extension of a function $u(t, x_1, \dots, x_n)$ from $W^{(1,2p,\dots,2p)}_{(t,x_1,\dots,x_n),2}(Q)$ to the whole space was carried out with preservation of the norm. In this case the domain Q in (^{5}) (if, for simplicity, it is assumed convex) is bounded by a smooth surface Γ, which with the tangent planes $t = t_B$ and

$t = t_H$, $t_H = \inf\{t,\ (x_1, \dots, x_n, t) \in Q\}$, $t_B = \sup\{t,\ (x_1, \dots, x_n, t) \in Q\}$, has contact of at least order $2p$.

In the present note the question is considered of extending, from a two-dimensional domain Q, a function $u(x, y)$ belonging to the space $W^{(m,n)}_{(x,y),2}(Q)$ (4) and satisfying certain boundary conditions, to the whole plane with preservation of the norm. For simplicity in the formulation of the result we assume the domain Q convex, and $m < n$ (the case $m = n$ was considered in (2,4)).

Theorem. Let $u(x, y) \in W^{(m,n)}_{(x,y),2}(Q)$, $u|_\Gamma = D_{xu}|_\Gamma = \cdots = D^{[m]-1}_x u|_\Gamma = 0$, where Γ is a sufficiently smooth boundary of the domain Q. If, in neighborhoods of the points $\Lambda = (x_\Lambda, y_\Lambda)$ and $\Pi = (x_\Pi, y_\Pi)$, $x_\Lambda = \inf\{x,\ (x, y) \in \Gamma\}$, $x_\Pi = \sup\{x,\ (x, y) \in \Gamma\}$, the equation Γ can be represented in the form $x - x_\Lambda = O(|y - y_\Lambda|^{n/m})$, $x_\Pi - x =$

$$= O\left(|y - y_\Pi|^{n/m}\right),$$

respectively, then there exists a function $u^*(x, y)$, which is an extension of the function $u(x, y)$ to the whole plane R and is such that

$$\|u^*\|_{W^{(m,n)}_{(x,y),2}(R)} \le C\|u\|_{W^{(m,n)}_{(x,y),2}(Q)}, \tag{2}$$

where C is a constant depending only on the domain Q.

Before proceeding to the proof of the theorem, which for brevity will be carried out for integer m and n, we formulate several auxiliary assertions concerning fractional derivatives. Let $f(x)$ be a function of one variable x of class $L_2(0, 1)$.

We shall say that $f(x)$ has a derivative $f^{(\alpha)}(x)$ of order α, $0 < \alpha < 1$, from $L_2(0, 1)$, if $f^{(\alpha)}(x) \in L_2(0, 1)$ and

$$\int_0^x f^{(\alpha)}(x)\,dx = \frac{1}{\Gamma(1-\alpha)} \int_0^x \frac{f(\xi)}{(x-\xi)^\alpha}\,d\xi. \tag{3}$$

Lemma 1. If the function $f(x) \in L_2(0, 1)$ has a derivative $f^{(\alpha)}(x)$ of order α, $0 < \alpha < 1$, from $L_2(0, 1)$, then

$$\int_0^x f(\xi)\,d\xi = \frac{1}{\Gamma(1+\alpha)} \int_0^x (x-\xi)^{(\alpha)} f^{(\alpha)}(\xi)\,d\xi. \tag{4}$$

Lemma 2. If the function $f(x) \in L_2(0, 1)$ has a derivative $f^{(\alpha)}(x)$, $0 < \alpha < 1$, from $L_2(0, 1)$, then

$$\left\|\frac{f}{x^\alpha}\right\|_{L_2(0,1)} \le C_1\|f^{(\alpha)}\|_{L_2(0,1)}, \tag{5}$$

where C_1 is an absolute constant.

Lemma 1 is proved analogously to the corresponding assertions on fractional derivatives in the book (6), and Lemma 2 can be obtained from Theorem 319 of the book (7).

Lemma 3. The set of functions $f(x) \in L_2(0,1)$ for which there exists a derivative of order α, $0 < \alpha < 1$, from $L_2(0,1)$, coincides with Slobodetskii' s space $W_2^{(\alpha)}(0,1)$.

Lemma 3 is proved by applying the Fourier transform to (3) and using the corresponding theorem of Slobodetskii (4).

Lemma 4. Let $u(x,y)$ be a finite function, concentrated in the domain Q and belonging to the space $W^{(m,n)}_{(x,y),2}(R)$, and let α and β be integers, $\alpha \geq 0$, $\beta \geq 0$, such that

$$1 - \frac{1}{n} \leq \frac{\alpha}{m} + \frac{\beta}{n} \leq 1.$$

Then

$$\left\| \frac{D_x^\alpha D_y^\beta u}{x^{m\left(1-\frac{\alpha}{m}-\frac{\beta}{n}\right)}} \right\|_{L_2(Q)} \leq C_2 \|u\|_{W^{(m,n)}_{(x,y),2}(Q)}, \tag{6}$$

where C_2 is a constant depending only on the domain Q.

For the proof of the theorem, cut the domain Q by the straight lines $x = x_\Lambda + \delta$ and $x = x_\Pi - \delta$ into the parts

$$Q_1^{(\delta)} = Q \cap (x_\Lambda \leq x \leq x_\Lambda + \delta), \quad Q_2^{(\delta)} = Q \cap (x_\Lambda + \delta < x < x_\Pi - \delta),$$

$$Q_3^{(\delta)} = Q \cap (x_\Pi - \delta \leq x \leq x_\Pi),$$

where δ is some positive sufficiently small number, $\delta < (x_\Pi - x_\Lambda)/2$. From the condition of the theorem it follows that the equations of the curves $\Gamma \cap (x_\Lambda \leq x \leq x_\Lambda + \delta)$ and $\Gamma \cap (x_\Pi - \delta \leq x \leq x_\Pi)$, parts of the boundary of $Q_1^{(\delta)}$ and $Q_3^{(\delta)}$ in neighborhoods of the points Λ and Π, respectively, can be represented in the form

$$x - x_\Lambda = \psi_\Lambda(|y - y_\Lambda|) \quad \text{and} \quad x_\Pi - x = \psi_\Pi(|y - y_\Pi|),$$

where

$$\psi_\Lambda(t) \leq A t^{n/m}, \quad \psi_\Pi(t) \leq A t^{n/m}, \quad 0 \leq t \leq \min\left(\psi_\Lambda^{-1}(\delta), \psi_\Pi^{-1}(\delta)\right),$$

and A is a constant independent of δ, provided δ is less than some sufficiently small δ_0.

Lemma 5. If the function $u(x,y)$ satisfies the conditions of the theorem, then for it there exists an extension from the domain $Q_2^{(\delta)}$ into the strip

$$R_2^{(\delta)} = (x_\Lambda + \delta < x < x_\Pi - \delta).$$

Moreover,

$$\|u^*\|_{W^{(m,n)}_{(x,y),2}(R_2^{(\delta)})} \leq C(\delta) \|u\|_{W^{(m,n)}_{(x,y),2}(Q_2^{(\delta)})}, \tag{7}$$

where the constant $C(\delta)$ does not depend on $u(x,y)$.

Lemma 5 is proved analogously to the corresponding theorems in $(^{2,4,8})$ by means of a local straightening of the boundary $\Gamma \cap (x_\Lambda + \delta < x < x_\Pi - \delta)$ of the domain $Q_2^{(\delta)}$. In extending $u(x,y)$ from the domain $Q_1^{(\delta)}$ (as well as from the domain $Q_3^{(\delta)}$) we shall assume that $x_\Lambda = y_\Lambda = 0$ and that the function $u(x,y)$, together with all its existing derivatives, vanishes on the line $x = 0$.

Define in $\widetilde{Q}_1^{(\delta)} = (0 \le x \le \delta,\ y \ge 0) \setminus Q_1^{(\delta)} \cap (y \ge 0)$ the function

$$v_1(x,y) = \sum_{i=1}^{N} \lambda_i e^{M_i(1-y/\varphi(x))}\, u(x, \varphi(x) e^{m_i(1-y/\varphi(x))}), \tag{8}$$

where $y = \varphi(x)$ is the equation $x = \psi_\Lambda(y)$, solved for y for $y > 0$; $m_i \ge 1,\ M_i \ge m_i,\ i = 1, \dots, N$, are certain positive numbers. It has been proved that, with a suitable choice of m_i and M_i, one can find numbers $N, \lambda_i,\ i = 1, \dots, N$, such that the function $v_1(x,y)$, defined in $\widetilde{Q}_1^{(\delta)}$, will coincide with the function $u(x,y)$, given in $Q_1^{(\delta)}$, along their common boundary $y = \varphi(x)$ with any number of derivatives with respect to x and to y. In the same way one constructs the function $v_2(x,y)$, which carries out the extension of $u(x,y)$ from $Q_1^{(\delta)}$ to the domain $\widetilde{Q}_1^{(\delta)} = (0 \le x \le \delta,\ y \le 0) \setminus Q_1^{(\delta)} \cap (y \le 0)$. Denoting by $u^*(x,y)$ the function defined in the strip $R_1^{(\delta)} = (0 \le x \le \delta)$, coinciding with $u(x,y)$ in $Q_1^{(\delta)}$, with $v_1(x,y)$ in $\widetilde{Q}_1^{(\delta)}$, and with $v_2(x,y)$ in $\widetilde{Q}_1^{(\delta)}$, and taking into account the regularity of the boundary Γ, from (8) we obtain

$$\|D_y^n u^*\|^2_{L_2(R_1^{(\delta)})} \le C_3^2 \sum_{k=0}^{n} \left\| \frac{D_y^k u}{x^{(n-k)m/n}} \right\|^2_{L_2(Q_1^{(\delta)})}, \tag{9}$$

$$\|D_x^m u^*\|^2_{L_2(R_1^{(\delta)})} \le C_3^2 \sum_{\substack{0 \le s/n + k/m \le 1 \\ k+s=m}} \sum \left\| \frac{D_x^k D_y^s u}{x^{m(1-k/m-s/n)}} \right\|^2_{L_2(Q_1^{(\delta)})}, \tag{10}$$

where the constant C_3, for sufficiently small δ, does not depend on δ. Analogous estimates are obtained:

$$\|D_y^s u^* \cdot \theta(x)\|^2_{L_2(R_1^{(\delta)})} \le C_3^2 \sum_{k=0}^{s} \left\| \frac{D_y^k u}{x^{(s-k)m/n}}\, \theta(x) \right\|^2_{L_2(Q_1^{(\delta)})}, \qquad 0 \le s \le n; \tag{9_s}$$

$$\|D_x^s u^* \cdot \theta(x)\|^2_{L_2(R_1^{(\delta)})} \leq C_3^2 \sum_{\substack{k_1+k_2=s \\ 0\leq k_1+k_2m/n\leq s}} \sum \left\| \frac{D_x^{k_1} D_y^{k_2} u \cdot \theta(x)}{x^{s-k_1-k_2m/n}} \right\|^2_{L_2(Q_1^{(\delta)})}, \qquad 0 \leq s \leq m, \tag{10_s}$$

for an arbitrary $\theta(x) \geq 0$ for which (9_s) and (10_s) make sense.

Let us note that the function $u^*(x, y)$ in $R_1^{(\delta)}$, for sufficiently large $|y|$, may be taken to be zero, without violating the estimates (9), (10), (9_s), and (10_s). Moreover, without violating these estimates, one may also regard $u^*(x, y)$ as being defined not only in $R_1^{(\delta)}$, but in the whole half-strip $x \leq x_\Lambda + \delta$, setting it equal to zero for $x \leq x_\Lambda$. This follows directly from (8), since $e^{M_i(1-y/\varphi(x))}$ for $x = 0,\ y \neq 0$ vanishes together with all its derivatives. In connection with this, the norms standing on the right-hand side of (9), for $k < n$, can be estimated as follows:

$$\left\| \frac{D_y^k u}{x^{(n-k)m/n}} \right\|^2_{L_2(Q_1^{(\delta)})} \leq \left\| \frac{D_y^k u^*}{x^{(n-k)m/n}} \right\|^2_{L_2(R_1^{(\delta)})} \leq$$

$$\leq \varepsilon \left\| \frac{D_y^{k+1} u^*}{x^{(n-k-1)m/n}} \right\|^2_{L_2(R_1^{(\delta)})} + \frac{1}{\varepsilon} \left\| \frac{D_y^{k-1} u^*}{x^{(n-k+1)m/n}} \right\|^2_{L_2(R_1^{(\delta)})}.$$

for any $\varepsilon > 0$. Hence, with the aid of (9_s), for a suitable choice of $\theta(x)$,

$$\left\| \frac{D_y^n u}{x^{(n-k)m/n}} \right\|^2_{L_2(Q_1^{(\delta)})} \leq \varepsilon \|D_y^n u^*\|^2_{L_2(R_1^{(\delta)})} + C_\varepsilon \left\| \frac{u}{x^m} \right\|^2_{L_2(Q_1^{(\delta)})}, \tag{11}$$

where C_ε depends only on $\varepsilon > 0$. From (11) and (9) it follows that

$$\|D_y^n u^*\|^2_{L_2(R_1^{(\delta)})} \leq B \left(\|D_y^n u\|^2_{L_2(Q_1^{(\delta)})} + \left\| \frac{u}{x^m} \right\|^2_{L_2(Q_1^{(\delta)})} \right), \tag{12}$$

where the constant $B > 0$.

Similarly, with the aid of (10_s), estimates are obtained for the terms on the right-hand side of (10) when

$$\frac{k}{m} + \frac{s}{n} > 1 - \frac{1}{n}:$$

$$\left\|\frac{D_x^k D_y^s u}{x^{m(1-k/m-s/n)}}\right\|^2_{L_2(Q_1^{(\delta)})} \leq \varepsilon \left\|\frac{D_x^k D_y^{s+\lambda} u}{x^{m(1-k/m-(s+\lambda)/n)}}\right\|^2_{L_2(R_1^{(\delta)})} + C_\varepsilon \left\|\frac{D_x^k u}{x^{m-k}}\right\|^2_{L_2(Q_1^{(\delta)})}, \tag{13}$$

where λ is such an integer that

$$1 - \frac{1}{n} \leq \frac{k}{m} + \frac{s+\lambda}{n} \leq 1.$$

Applying Lemmas 3 and 4 to the remaining terms of the right-hand side of (10), as well as to the first term of (13), we finally obtain

$$\|D_x^m u\|^2_{L_2(R_1^{(\delta)})} \leq B \left(\|D_y^n u\|^2_{L_2(Q_1^{(\delta)})} + \|D_x^m u\|^2_{L_2(Q_1^{(\delta)})} + \sum_{r=0}^{m} \left\|\frac{D_x^r u}{x^{m-r}}\right\|^2_{L_2(Q_1^{(\delta)})} \right). \tag{14}$$

By means of integration by parts with respect to x over the domain $Q_1^{(\delta)}$, one can prove the existence of a constant C_4, independent of $u(x,y)$, such that

$$\left\|\frac{D_x^r u}{x^{m-r}}\right\|^2_{L_2(Q_1^{(\delta)})} \leq C_4^2 \|D^m u\|^2_{L_2(Q_1^{(\delta)})}. \tag{15}$$

Taking into account (15), (14), (12), and (10), we obtain

$$\|u\|^2_{W^{(m,n)}_{(x,y),2}(R_1^{(\delta)})} \leq C_5^2 \|u\|^2_{W^{(m,n)}_{(x,y),2}(Q_1^{(\delta)})}, \tag{16}$$

with a constant C_5 independent of $u(x,y)$ and of δ for $\delta < \delta_0$.

Since estimate (16) remains valid if in it the domains $Q_1^{(\delta)}$ and $R_1^{(\delta)}$ are replaced by $Q_3^{(\delta)}$ and $R_3^{(\delta)}$, respectively, it follows immediately from (16) and Lemma 5, in which δ should be replaced by δ_0, that the assertion of the theorem is obtained.

Remark. From this theorem, with the aid, for example, of (4) or (8), there immediately follows the existence in Q of the generalized derivatives $D_x^\mu D_y^\nu u \in L_2(Q)$ for

$$\frac{\mu}{m} + \frac{\nu}{n} \leq 1$$

and

$$\|D_x^\mu D_y^\nu u\|_{L_2(Q)} \leq C_1 \|u\|_{W^{(m,n)}_{(x,y),2}(Q)}, \qquad \frac{\mu}{m} + \frac{\nu}{n} \leq 1,$$

where C_1 does not depend on $u(x, y)$. Moreover, in Q the corresponding results proved for R_2 on the behavior of $u(x, y)$ on manifolds of a smaller number of dimensions (on curves in the domain Q) turn out to be valid.

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Note: Figure translations are in progress. See original paper for figures.