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L. I. Kamynin

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Abstract

Full Text

L. I. Kamynin

ON A PROBLEM IN HYDRAULIC ENGINEERING

(Presented by Academician S. L. Sobolev on 25 VII 1961)

The note considers the solution of a mixed problem for a parabolic equation with coefficients that have a discontinuity along a moving and initially unknown line. Problems of this kind arise in the practice of hydraulic engineering, for example (see (^{1})) in the injection of binding solutions into rock formations to increase the strength and water impermeability of the foundations of hydraulic structures. If injection is carried out simultaneously from a number of wells located on one straight line at equal and sufficiently small distances from one another, then, in order to determine the pressure $u_1(x,t)$ of the injected liquid and the pressure $u_2(x,t)$ of the displaced liquid, moving in a porous medium without mixing (where $x = h(t)$ is the line of their interface), one may formulate the following mathematical problem, which in what follows we shall call Verigin' s problem.

It is required to find three functions $u_1(x,t)$, $u_2(x,t)$, and $h(t)$ satisfying the system of equations of parabolic type

$$\frac{\partial u_1}{\partial t} = \frac{\partial}{\partial x}\left(a_1(x,t)\frac{\partial u_1}{\partial x}\right), \qquad 0 < x < h(t);$$

$$\frac{\partial u_2}{\partial t} = \frac{\partial}{\partial x}\left(a_2(x,t)\frac{\partial u_2}{\partial x}\right), \qquad h(t) < x < l;\ 0 < t < T; \tag{1}$$

the initial conditions

$$u_1(x,0) = \psi_1(x), \qquad 0 \le x \le h(0) = c;$$

$$u_2(x,0) = \psi_2(x), \qquad c \le x \le l; \tag{2}$$

the boundary conditions

$$\frac{\partial u_1(0,t)}{\partial x} = \varphi_1(t);$$

$$\frac{\partial u_2(l,t)}{\partial x} = \varphi_2(t), \qquad 0 \le t \le T; \tag{3}$$

the conjugation conditions on the unknown line of discontinuity of the coefficients $x = h(t)$

$$u_1(h(t), t) = u_2(h(t), t);$$

$$a_1(h(t), t)\frac{\partial u_1}{\partial x}(h(t), t) = a_2(h(t), t)\frac{\partial u_2}{\partial x}(h(t), t), \qquad 0 \le t \le T, \tag{4}$$

where, moreover, for the line of discontinuity the differential equation must hold

$$\frac{dh(t)}{dt} = -\beta(h(t), t)\frac{\partial u_1}{\partial x}(h(t), t), \tag{5}$$

where

$$\beta(x, t) = B\,\frac{\alpha_1(x, t)a_2(x, t) - a_2(x, t)a_1(x, t)}{a_2(x, t)}$$

$$(B > 0 \text{ is a constant}).$$

In addition, the compatibility conditions are satisfied

$$\begin{gathered} \psi_1'(0) = \varphi_1(0); \\ \psi_2'(l) = \varphi_2(l); \\ \psi_1(c) = \psi_2(c); \\ \alpha_1(c, 0)\psi_1'(c) = \alpha_2(c, 0)\psi_2'(c). \end{gathered} \tag{6}$$

With regard to the coefficients of equation (1), the initial functions (2), and the boundary functions (3)–(5), we shall henceforth assume that the following conditions are fulfilled:

I. The functions $a_i(x, t)$ are continuous in the closed domain

$$\overline{G}_T = \{(x, t);\ 0 \le x \le l,\ 0 \le t \le T\}$$

together with the derivatives $\partial a_i(x, t)/\partial x$, $\partial a_i(x, t)/\partial t$, and $\partial^2 a_i(x, t)/\partial x^2$ $(i = 1, 2)$, and $a_i(x, t)$, $\partial a_i(x, t)/\partial x$, and $\partial a_i(x, t)/\partial t$ satisfy a Hölder condition in x and t with nonzero exponent; moreover,

$$0 < a_i \le a_i(x, t) \le A_i,$$

where a_i, A_i $(i = 1, 2)$ are constants.

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II. The functions $\varphi_i(t)$ $(i = 1, 2)$ satisfy a Hölder condition in t with nonzero exponent; the functions $\psi_i(x)$ $(i = 1, 2)$ are continuous together with $\psi_i'(x)$, and $\psi_i'(x)$ satisfy a Hölder condition in x with nonzero exponent; moreover, $\varphi_i(t) < 0$ and $\psi_i'(x) < 0$ $(i = 1, 2)$.

III. In the domain $\overline{G}_T$ the functions $a_i(x, t)$ $(i = 1, 2)$ satisfy a Hölder condition in x and t with nonzero exponent, and

$$\sqrt{a_1(x,t)}\,a_2(x,t) + \sqrt{a_2(x,t)}\,a_1(x,t) \neq 0,$$

$\alpha_i(x, t) > 0$ $(i = 1, 2)$, $A \geq \beta(x, t) \geq a > 0$, where A and a are constants.

IV. The condition $0 < c < l$ is fulfilled.

When conditions I–IV are fulfilled, the existence of a solution of the Verigin problem (1)–(6) is proved. We note that, in the case of constant $a_i(x, t)$ and $\alpha_i(x, t)$, the Verigin problem (1)–(6) was considered by N. N. Verigin (^{1}) and L. I. Rubinshtein (^{2}). We outline the scheme of the proof.

Assuming that $h(t)$ is a monotonically nondecreasing function, by integrating equations (1) over the domains

$$\overline{G}_T^{(1)} = \{(x,t),\ 0 \leq x \leq h(t),\ 0 \leq t \leq T\}$$

and

$$\overline{G}_T^{(2)} = \{(x,t),\ h(t) \leq x \leq l;\ 0 \leq t \leq T\},$$

respectively, and by using the initial and boundary conditions (2)–(4), one can obtain an integral equation for determining $h(t)$ (cf., in this connection, the method of the authors who dealt with the solution of the Stefan problem ($^{6-9}$)):

$$h(t) = Sh(t) \equiv c + B\left\{ \int_0^{h(t)} u_1(x,t;h)\,dx + \int_{h(t)}^{l} u_2(x,t;h)\,dx \right.$$
$$\left. + \int_0^t \left[a_1(0,\tau)\varphi_1(\tau) - a_2(l,\tau)\varphi_2(\tau)\right] d\tau - \int_0^c \psi_1(x)\,dx - \int_c^l \psi_2(x)\,dx \right\}. \tag{7}$$

Consider the set $\overline{M}$ obtained as the closure, in the metric of C, of the set of continuously differentiable functions $h(t)$ $(0 \leq t \leq T)$ for which $h(0) = c$ and $0 \leq dh(t)/dt \leq L$. On the set $\overline{M}$ we consider the operator Sh, $g = Sh$, defined by means of the right-hand side of (7), where $u_i(x, t; h)$ $(i = 1, 2)$ is the solution of the auxiliary problem (1)–(4), (6) with the given function $h(t)$ from $\overline{M}$. As was proved by the author in (^{3}), the problem (1)–(4), (6) has a solution $u_i(x, t; h)$ $(i = 1, 2)$, continuous together with

$$\frac{\partial u_i}{\partial x}(x,t;h)$$

on the closure of $\overline{G}_T^{(i)}$, provided conditions I—IV are fulfilled, if only $h(t)$ satisfies a Hölder condition in t with exponent $> 1/2$, which holds in our case, since $h(t) \in \overline{M}$ satisfies a Hölder condition

with exponent 1. The following lemmas are needed for the proof of the existence theorem.

Lemma 1. If $u_i(x, t, h)$ is a solution of the auxiliary problem (1)—(4), (6), then, under conditions I—IV, there exists a constant D, independent of the choice of $h(t)$ from M, such that in $\bar{G}_T^{(i)}$

$$-D \leqslant \frac{\partial u_i}{\partial x}(x, t; h) \leqslant 0.$$

Lemma 2. If $h(t)$ is a monotonically nonincreasing function satisfying a Hölder condition in t with exponent 1, then $g(t) = Sh(t)$ will be a differentiable monotonically nonincreasing function for which

$$g(0) = c, \qquad 0 \leqslant \frac{dg(t)}{dt} \leqslant D \max_{\bar{G}_T} \beta(x, t) = L.$$

Using the theorem on the continuous dependence of the solution of problem (1) —(4), (6) on the boundary, contained in our paper (^{4}), one proves

Lemma 3. The operator Sh is continuous in the norm

$$\|g\| = \max_{0 \leqslant t \leqslant T} |g(t)| + \sup_{0 \leqslant t_1, t_2 \leqslant T} \frac{|g(t_1) - g(t_2)|}{|t_1 - t_2|}$$

on the set $\overline{M}$.

Using the results of our paper (^{3}) and of the paper of Gevrey (^{5}), one proves

Lemma 4. There exists at least one fixed point of the mapping $g = Sh$ of the set $\overline{M}$ into itself.

Finally, with the aid of Lemma 4 and the integral equation (7), the main theorem is proved:

Theorem. Under conditions I—IV, the Verigin problem (1)—(6) has at least one solution $\{u_1(x, t; h);\ u_2(x, t; h);\ h(t)\}$, and both

$$u_i(x, t; h), \quad \text{as well as} \quad \frac{\partial u_i}{\partial x}(x, t; h)$$

satisfy a Hölder condition in x and t in $\bar{G}_T^{(i)}$ with a nonzero exponent.

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Moscow State University
named after M. V. Lomonosov

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