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Abstract

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THEORY OF ELASTICITY

A. L. KRYLOV

JUSTIFICATION OF DIRICHLET' S PRINCIPLE FOR THE FIRST BOUNDARY-VALUE PROBLEM OF THE NONLINEAR THEORY OF ELASTICITY

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1. It is known (see, for example, ([1], [2])) that the equilibrium equations of an elastoplastic medium can be derived from certain variational principles, consisting in the fact that the solution of the problem gives a minimum to certain positive definite functionals. The fact of the existence of a solution of these variational problems is called Dirichlet' s principle and is widely used in heuristic arguments. Dirichlet' s principle for certain linear equations was proved in a number of works by R. Courant ([3]), S. L. Sobolev ([4], [8]); finally, in 1942 it was obtained by K. Friedrichs ([5]) for the first and second boundary-value problems of the theory of elasticity and by D. M. Eidus ([6]) for the third and fourth problems (for more details see ([7])). W. T. Koiter posed the problem of justifying Dirichlet' s principle in nonlinear problems of the theory of elastoplastic media (([1]), § 7); in the present work this problem is solved for the case when displacements are prescribed on the boundary (the first boundary-value problem).

Variational principles for certain two-dimensional problems of nonlinear elasticity theory were justified by A. Langenbach ([13,14]), whose methods are partly analogous to those we use in Sec. 2 (see also ([10])). (Langenbach considers, in particular, the same example of the torsion problem, as well as a number of other problems.) Our principal aim, however, was the consideration of the general equations of the theory of elasticity.

2*. By a nonlinear elastic medium we shall mean a medium in which the deviators of the strain tensors D_ε (rates of strain) and stresses D_σ are connected by a relation of the form

$$D_\varepsilon = \psi D_\sigma, \tag{1}$$

where $\psi = f(T)$, T is the intensity of the shear stresses (in this case one sometimes speaks of a medium with hardening). For $\psi = 1/2G$ (constant) we have ordinary linear elasticity (Hooke' s law).

The problem that we shall consider is formulated as follows: find the displacement vector

$$u(x) = \{u_1(x_1, x_2, x_3),\, u_2(x_1, x_2, x_3),\, u_3(x_1, x_2, x_3)\},$$

defined in a smooth domain Ω, assuming a prescribed function φ on the boundary of the domain $D\Omega$, and giving a minimal value to the integral

$$\Pi(u) = \iiint_\Omega \left(\frac{\varepsilon^2}{6k} + \int_0^{\Gamma^2} g(\zeta)\, d\zeta \right) d\Omega, \qquad (*)$$

where k is a constant; ε and Γ^2 are the linear and quadratic invariants of the strain tensor $\|\varepsilon_{ik}\|$ and of the deviator of the strain tensor $\|\varepsilon_{ik} - \delta_{ik}\varepsilon\|$, $g(\Gamma)$ is a function determined from $f(T^2)$ by the equalities $f(T^2) = \frac{1}{2}T\Gamma$, $T = 2g(\Gamma^2)\Gamma$. The components of the strain tensor ε_{ik} are related to the displacement vector u_i by the formulas

$$\varepsilon_{ik} = \frac{1}{2}\,(u_{i,k} + u_{k,i}),$$

where $u_{ik} = \partial u_i(x_1, x_2, x_3)/\partial x_k$, $i, k = 1, 2, 3$. The function $g(\Gamma^2)$ can often be represented in the form $g(\Gamma^2) = B(\Gamma^2)^{(\beta-2)/2}$, where $1 \ll \beta \ll 2$.

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* See (2), Chapters I-III (especially §§ 18 and 33). See the notation there as well.

Thus, $\Pi(u)$ is a functional depending, generally speaking, non-quadratically on linear combinations of the first derivatives of the solution, namely on $\varepsilon_{ik} = \frac{1}{2}(u_{i,k} + u_{k,i})$. From the form of the function $g(\Gamma^2)$ (2) it follows that $\Pi(u)$ is a uniformly convex functional of order $\beta \leqslant 2$ in a sense that will be explained in §§ 3 and 4. For $\beta = 2$ we have the linear case, considered in (7). The minimum is sought among functions for which the integral is meaningful.

3. To make the solution of this problem easier to understand, we shall first examine a simpler example, in which we shall clearly reveal the method of our reasoning. Namely, consider the problem of finding a function $F(x, y)$ giving a minimum to the functional

$$I(u) = \iint_\Omega \left((F_x^2 + F_y^2)^{p/2} - 2\omega F \right) d\Omega, \qquad (2)$$

satisfying the boundary condition

$$F(x,y)\big|_{\partial\Omega} = 0. \tag{3}$$

(This is the problem of torsion of a prism with hardening; here $F(x,y)$ is the stress function, ω is the twist per unit length, and $p \geqslant 2$ is a constant ($p = 2$ is the linear case), see ([2]), § 44.) The minimum is sought in the class of functions having first derivatives (in the Sobolev sense), summable to the power p, and vanishing on the boundary; we shall denote the class of these functions by $\overset{\circ}{W}{}_p^{(1)}$ (see ([8])). In the further arguments we shall follow our note ([9]) (for more detail see ([10])). Using the embedding theorems ([8]), let us note that for any $u(x,y) \in \overset{\circ}{W}{}_p^{(1)}$

$$\|u\|_{L_q} \leqslant \|u\|_{\overset{\circ}{W}{}_p^{(1)}} \qquad (q \geqslant 1 \text{ arbitrary}), \tag{4}$$

where

$$\|u\|_{L_q} = \left\{\iint_\Omega |u|^q \, d\Omega\right\}^{1/q}, \qquad \|u\|_{\overset{\circ}{W}{}_p^{(1)}} = \left(\iint_\Omega (u_x^2 + u_y^2)^{p/2} \, d\Omega\right)^{1/p}, \tag{5}$$

and, moreover, for $p > 2$

$$\|u\|_C \leqslant C\|u\|_{\overset{\circ}{W}{}_p^{(1)}}. \tag{6}$$

We at once obtain that $I(u) \geqslant \text{const}$ on $\overset{\circ}{W}{}_p^{(1)}$, i.e. it is bounded below. Therefore there exists a lower bound d of $I(u)$ on $\overset{\circ}{W}{}_p^{(1)}$ and a sequence of functions $\{F^n\} \in \overset{\circ}{W}{}_p^{(1)}$ (called a minimizing sequence) such that

$$I(F^n) \searrow d. \tag{7}$$

To prove the Dirichlet principle for the given problem means to prove that this sequence converges to some function $F^*(x,y) \in \overset{\circ}{W}{}_p^{(1)}$, and that $I(F^*) = d$. Our functional $I(F)$ has the important property of uniform convexity, which consists in the following: if $I(u) < d+\varepsilon$, $I(v) < d+\varepsilon$, and $I(\frac{u+v}{2}) > d$, then $I(u-v) \to 0$ as $\varepsilon \to 0$. This property follows from Clarkson' s first inequality ([8])

$$\left\|\frac{f+g}{2}\right\|_{L_p}^p + \left\|\frac{f-g}{2}\right\|_{L_p}^p \leqslant \frac{1}{2}\left(\|f\|_{L_p}^p + \|g\|_{L_p}^p\right) \qquad (p \geqslant 2). \tag{8}$$

If the convexity criterion is applied to the terms of the minimizing sequence, then we obtain that $I(F^m - F^n) \to 0$ as $m, n \to \infty$.

Since the sequence $\{F^m\}$ is bounded in $\overset{0}{W}{}_p^{(1)}$, one can choose from it a subsequence such that $\{F^m\}$ will converge in L_q (q arbitrary). Then the term $\int_\Omega \omega(F^m - F^n)\, d\Omega \to 0$, and we obtain that

$$\|F^m - F^n\|_{\overset{0}{W}{}_p^{(1)}} \to 0,$$

i.e., the sequence is fundamental in $\overset{0}{W}{}_p^{(1)}$. By completeness of $\overset{0}{W}{}_p^{(1)}$ ([8])

$$F^n(x, y) \to F^*(x, y) \in \overset{0}{W}{}_p^{(1)},$$

with $I(F^*) = d$. Let us add one more remark on the boundary conditions. For $p = 2$ the problem admits only a one-dimensional boundary, since convergence in this case is weaker than uniform convergence (for details see ([8]); whereas in the case $p > 2$ the boundary may consist of a one-dimensional part Γ_1 and a zero-dimensional part Γ_0 (i.e., individual points). The function F^* will be discontinuous in such a domain Ω (for example, in a disk with the center removed). Of course, the stresses, which are derivatives of the function $F^*(x, y)$, will have infinite values at these removed points. However, one can consider a sequence of annuli with decreasing inner radius. Then for $p = 2$ the solutions will converge to the solution of the problem on the full disk (without the cut-out), while for $p > 2$ the solution of the limiting problem will differ from the solution of the problem without the cut-out. Thus a linear membrane "breaks" under point fastening, whereas a nonlinear one (for $p > 2$) withstands it.

4. Let us now consider the problem of the minimum of the integral (*). Since $\Pi(u) \geq 0$, there exists a minimizing sequence $u^n \in \overset{\varphi}{W}{}_\beta^{(1)}$ of functions for which

$$\Pi(u^n) \searrow d.$$

From the properties of uniform convexity of $\Pi(u)$, which for $\beta \leq 2$ follows from the second Clarkson inequality ([8]), analogously to item 3 we obtain

$$\Pi(u^n - u^m) \to 0 \quad \text{as } m, n \to \infty.$$

Unlike the preceding considerations, it is impossible from this to draw directly a conclusion about the convergence of $\{u^n\}$ in the norm $W_\beta^{(1)}$, since the functional $\Pi(v)$ contains powers not of the derivatives $v_{i,k}$ themselves, but of their linear combinations $\varepsilon_{ik} = \frac{1}{2}(v_{i,k} + v_{k,i})$. An analogous situation also occurs in the linear case, where it is overcome by proving Korn' s inequality ([5,7])

$$\Pi(v) \geq \|v\|_{\overset{0}{W}{}_\beta^{(1)}} \tag{9}$$

(of course, for the case $\beta = 2$). We shall prove inequality (9) for all $\beta > 1$.

It is easy to show that from $\Pi(u) \to 0$ it follows that

$$\int_\Omega \left[\sum_{i,k=1}^{3} \varepsilon_{ik}^2\right]^{\beta/2} d\Omega \to 0;$$

thus, in order to prove (9), it is sufficient to show that

$$\|v\|_{\overset{0}{W}{}_\beta^{(1)}} \leq C \int_\Omega \left[\sum \varepsilon_{ik}^2\right]^{\beta/2} d\Omega \tag{10}$$

(where $\|v\|_{\overset{0}{W}{}_\beta^{(1)}} = \sum_{i=1}^{3} \|v_i\|_{\overset{0}{W}{}_\beta^{(1)}}$).

Thus, we have to estimate the norms of v_i in $\overset{0}{W}{}_\beta^{(1)}$ in terms of the norms of $\varepsilon_{ik} = v_{i,k} + v_{k,i}$ in L_β. Consider, for example, v_1. We have

$$\varepsilon_{11} = v_{1,1}; \qquad \varepsilon_{12} = \tfrac{1}{2}\left(v_{1,2} + v_{2,1}\right); \qquad \varepsilon_{13} = \tfrac{1}{2}\left(v_{1,3} + v_{3,1}\right),$$

and so on. Hence it is easy to obtain that v_1 satisfies the Poisson equation

$$\Delta v_1 = h, \qquad v_1|_{\gamma\Omega} = 0$$

with right-hand side belonging to $W_\beta^{(-1)}$, i.e., being a generalized function (the derivative of a function from L_β). Such a problem has a solution ([11]), obtained from the work of Calderon and Zygmund ([12]), for which the estimate

$$\|v\|_{\overset{0}{W}{}_\beta^{(1)}} \leq C \left\| \sum_{i,k=1}^{3} |\varepsilon_{ik}| \right\|_{L_p}.$$

is valid. Hence follows the validity of (10), and therefore of Korn' s generalized inequality (9). After this we obtain the convergence of $\{u^n\}$ in $\overset{\varphi}{W}{}_\beta^{(1)}$, and the Dirichlet principle for (*) is proved analogously to § 3.

Apparently, the proposed method can also be applied to the case of other boundary-value problems of the nonlinear theory of elasticity.

Moscow State University
named after M. V. Lomonosov

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Note: Figure translations are in progress. See original paper for figures.