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Abstract

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TEMPERATURE SEPARATION IN FREE-MOLECULAR EXPANSION INTO VACUUM

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A necessary condition for the free-molecular flow of a gas in a space of characteristic size l, where the minimum mean free path is λ, is the fulfillment of the inequality $l \ll \lambda$. We shall consider the simplest type of such a flow, when there is no interaction at all between the gas particles. The initial distribution of particle velocities at each point of the initial volume may, generally speaking, be arbitrary. Usually, however, one proceeds from the assumption that at the initial instant there exists local or overall thermodynamic equilibrium, which is equivalent to the existence of the corresponding Maxwellian velocity distribution.

The idea of calculating such a flow reduces to the following: in order for a particle that was at the point r_0 at the initial instant to arrive at the point r at time t, its velocity must be equal to $v = (r - r_0)/t$; on the other hand, the density of particles possessing such a velocity is proportional to the expression

$$n_0(r_0)\exp\{-mv^2/[2kT_0(r_0)]\}\,dv,$$

where $n_0(r_0)$ and $T_0(r_0)$ are, respectively, the initial particle density and temperature, while the remaining notation is conventional; as a result, using the relation of v to r_0 and t and integrating the above expression with the corresponding normalization coefficient over the initial volume, we obtain the desired particle density as a function of r and t. The analogy with the theory of heat conduction is obvious and is pointed out in ([1]).

In ([1,2]) the simplest problems of this type were solved, and it was shown that the results agree well with the solutions of the corresponding problems of gas dynamics. This latter circumstance is of unquestionable interest. An analogous one-dimensional problem for a bounded space was investigated in ([3]). The problem with a source in a special formulation was considered in ([4]).

In free-molecular expansion into vacuum, the "hotter" particles from the "tails" of the Maxwell distribution will move ahead. However, there are few such particles, and the mean temperature outside the initial volume, at first glance, will be the same as in the initial volume. We shall show here that in fact temperature separation does exist in the flow under investigation.

For simplicity we shall assume that a gas with constant particle density n_0 and identical particle mass m is, at the initial instant $t = 0$, in complete thermodynamic equilibrium at constant temperature T_0. Free-molecular expansion of the gas begins at $t = 0$ from centrally symmetric bodies of radius r_0 into the surrounding infinite space–vacuum. (The solution of the problem of filling a void is readily obtained from the solution of our problem.) We shall use the usual definitions of the energy density e and temperature T.

Introduce the dimensionless variables

$$\rho = r/r_0, \qquad \tau = t/t_0, \qquad \nu = n/n_0, \qquad \varepsilon = e/(n_0 k T_0),$$

where $t_0 = r_0[m/(2kT_0)]^{1/2}$, and k is Boltzmann' s constant.

Then

$$\nu(\rho,\tau) = \pi^{-3/2} \int \nu(\rho_1, 0) \exp\left[-\left(\frac{\rho-\rho_1}{\tau}\right)^2\right] \frac{d\rho_1}{\tau^3} \tag{1}$$

and the integration is performed over the initial volume.

For the dimensionless mean temperatures in the initial volume and outside it,

$$\theta_i = \int \varepsilon \, d\rho_i \Big/ \left({}^3/2 \int \nu \, d\rho_i\right) = \langle T_i \rangle / T_0 \quad (i = 1, 2),$$

we have

$$\theta_i = 1 + \frac{1}{3} \frac{d \ln \mu_i}{d \ln \tau}, \tag{2}$$

where $\mu_i = \int \nu \, d\rho_i$. In our particular case the normalization conditions are

$$\mu_1 + \mu_2 = 1, \ \pi, \ 4\pi/3$$

respectively in the plane, cylindrical, and spherical cases ($\nu = 1, 2, 3$).

Since in the present formulation of the problem there is no directed motion of the gas as a whole ($\langle v \rangle = 0$), the definition of mean temperature adopted by us as the mean kinetic energy of the translational motion of the particles coincides with the definition of temperature in kinetic theory. At equilibrium, as is not hard to see, our temperature coincides with the ordinary equilibrium temperature.

By direct calculation it is not difficult to obtain

$$\mu_1 = \frac{1}{2}\int_{-1}^{1} \operatorname{erf}\frac{\rho+1}{\tau}\,d\rho = \operatorname{erf} x - \frac{1}{\sqrt{\pi}x}\left(1-e^{-x^2}\right) \quad (\nu=1); \tag{3}$$

$$\begin{aligned}\mu_1 &= 2\pi\int_{-1}^{1} \operatorname{erf}\frac{\rho+1}{\tau}\rho^2\,d\rho + 2\sqrt{\pi}\tau\int_{-1}^{1}\exp\left[-\left(\frac{\rho+1}{\tau}\right)^2\right]\rho\,d\rho \\ &= \frac{4\pi}{3}\left\{\operatorname{erf} x + \frac{1}{\sqrt{\pi}x^3}\left[(x^2-2)e^{-x^2} - (3x^2-2)\right]\right\} \quad (\nu=3),\end{aligned} \tag{4}$$

where $\operatorname{erf} x = \frac{2}{\sqrt{\pi}}\int_0^x e^{-\eta^2}\,d\eta$ is the error integral and $x = 2/\tau$.

To obtain a compact expression in the cylindrical case, we differentiate the initial expression

$$\mu_1 = 2\int_0^1\int_0^1 \exp\left[-\frac{\rho^2+\rho_1^2}{\tau^2}\right] I_0\left(\frac{2\rho\rho_1}{\tau^2}\right)\frac{\rho_1\,d\rho_1}{\tau^2}\,2\pi\rho\,d\rho = 2\pi\frac{e^{-y}}{y}\sum_{k=0}^{\infty}\left[\sum_{l=k+1}^{\infty}\frac{y^l}{2^l l!}\right]^2$$

with respect to $y = 2/\tau^2$, and, using the relation

$$\sum_{k=1}^{\infty} I_k(x) = \frac{1}{2}\left[e^x - I_0(x)\right],$$

which is derived by applying the Laplace transform, we obtain

$$d\mu_1/dy = -\left[(\mu_1-\pi) + \pi e^{-y} I_0(y)\right]/y.$$

Integrating the latter expression, we finally obtain

$$\mu_1 = \pi\left[1 - \int_0^y e^{-\eta} I_0(\eta)\,d\eta\Big/ y\right] \quad (\nu=2). \tag{5}$$

Here $I_k(x)$ are Bessel functions of imaginary argument of order k.

In calculations the relation

$$\int_0^\infty e^{-\eta} I_0(2\sqrt{x\eta})\,d\eta = e^x$$

is useful.

The expressions θ_i are obtained from definition (2) and the expressions μ_1 (3)-(5). Table 1 gives the expansions of θ_i ($\nu = 1,2,3$) as $\tau \to 0+0$, $\tau \to \infty$. The

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curves θ_i ($\nu = 1, 2, 3$) as functions of τ are given in Fig. 1. In calculating μ_1 ($\nu = 2$) from (5), tables from [5] were used.

We see that θ_i are monotonically decreasing functions of τ. The maximum value of θ_2 is reached discontinuously at the point $\tau = 0$ and does not depend on the configuration of the initial volume. The minimum value of θ_2, naturally, coincides with the initial value of θ_1. The minimum value of θ_1 depends on the configuration of the initial volume, decreasing as its symmetry increases. A judgment about the rates of change of θ_i in the limiting cases can be obtained from Table 1.

Fig. 1. Dependence of the dimensionless temperatures inside and outside the initial volume $\theta_{1,2}$ on dimensionless time τ in the planar, cylindrical, and spherical cases ($\nu = 1, 2, 3$).

The meaning of the particular limiting values of θ_i can be interpreted as follows. Instantaneous expansion of an ideal gas into a vacuum leads, outside the initial volume, to an apparent increase in the number of degrees of freedom of the particles due, apparently, to the asymmetry of the expansion. The additional degree of freedom, while retaining the usual definition of temperature, leads to an increase of the latter by a factor of 4/3, independently of the configurations of the initial and final volumes. The order of symmetry of the initial volume

Table 1

		$\tau \to 0 + 0$	$\tau \to \infty$
$\nu = 1$	θ_1	$1 - (6\sqrt{\pi})^{-1}\tau + \cdots$	$\frac{2}{3} + \frac{4}{9}\tau^{-2} - \cdots$
$\nu = 1$	θ_2	$\frac{4}{3} - \frac{1}{3}\exp(-4\tau^{-2}) + \cdots$	$1 + 2(3\sqrt{\pi})^{-1}\tau^{-1} - \cdots$
$\nu = 2$	θ_1	$1 - (3\sqrt{\pi})^{-1}\tau + \cdots$	$\frac{1}{3} + \frac{2}{3}\tau^{-2} - \cdots$
$\nu = 2$	θ_2	$\frac{4}{3} - \frac{1}{24}\tau^2 + \cdots$	$1 + \frac{2}{3}\tau^{-2} - \cdots$
$\nu = 3$	θ_1	$1 - (2\sqrt{\pi})^{-1}\tau + \cdots$	$\frac{4}{5}\tau^{-2} - \cdots$
$\nu = 3$	θ_2	$\frac{4}{3} - \frac{1}{9}\tau^2 + \cdots$	$1 + 4(3\sqrt{\pi})^{-1}\tau^{-3} - \cdots$

determines the number of degrees of freedom of the gas particles remaining in it as $\tau \to \infty$; for $\nu = 1, 2, 3$ this number is respectively equal to $2, 1, 0$, and θ_1 decreases to $\frac{2}{3}, \frac{1}{3}, 0$ of its initial value.

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Note: Figure translations are in progress. See original paper for figures.