ON THE PROBABILISTIC FORMULATION OF TWO ECONOMIC PROBLEMS

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Abstract

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MATHEMATICS

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ON THE PROBABILISTIC FORMULATION OF TWO ECONOMIC PROBLEMS

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1. In the **traveling-salesman problem** it is required to find the shortest closed path $s(n)$ connecting n given points in r-dimensional Euclidean space. We shall regard these points as n independent realizations of an r-dimensional random variable with some distribution function

$$F(x) = F(x_1, x_2, \dots, x_r).$$

In the present note we construct rather simply arranged random variables estimating $s(n)$ (the path and its length will be denoted in the same way). It turns out, in particular, that for large n and under certain restrictions on $F(x)$ one can indicate a simple rule for constructing a path $s_1(n)$ such that the ratio $s_1(n)/s(n)$ "almost always" does not exceed $1+\rho$, $\rho > 0$. An estimate for the number ρ is contained in the work.

A. Lower estimate. Everywhere below we shall assume that the distribution function $F(x)$ has a continuous density $f(x)$. In ([2]) it is shown that with probability 1 there exists

$$\lim_{n\to\infty} n^{1/r-1} s(n) = M(f) = \mu_r \int f^{1-1/r}(x)\,dx; \qquad (1)$$

$$\mu_r \geq \frac{r\Gamma(3+1/r)\Gamma^{1/r}(1+r/2)}{2\sqrt{\pi}(r+1)} \qquad (r \geq 2). \qquad (2)$$

More effective methods for estimating $s(n)$ may be sought in the use of Chebyshev's inequality and in the construction of such a random variable $s_2(n) \leq s(n)$, sufficiently close to $s(n)$, for which it is possible to find the asymptotic behavior of $\mathbf{M}s_2(n)$ and $\mathbf{D}s_2(n)$. The following assertion, for example, belongs to this direction.

One can indicate a random variable $s_2(n) \leq s(n)$ for which

$$\lim_{n\to\infty} n^{1/r-1}\mathbf{M}s_2(n) = M_2(f), \qquad \lim_{n\to\infty} n^{2/r-1}\mathbf{D}s_2(n) = D_2(f),$$

where the functionals $M_2(f) > 0,\ D_2(f) > 0$ have been found. In particular,

$$M_2(f) = \sup_{\alpha} \left\{ \alpha^{1-r} \int \left[1 - \exp\left(-\alpha^r \left(\frac{r+1}{2r+1} \right)^r f(x) \right) \right] dx \right\}.$$

The estimate $s_2(n)$ can be improved.

B. Upper estimate. Here we shall also indicate a method for constructing an estimating path $s_1(n)$. Suppose additionally that the domain G in which $f(x) > 0$ is bounded, simply connected, and has a sufficiently smooth boundary. Consider the system of planes

$$x_s = k\alpha n^{-1/r}, \qquad s = 2, \dots, r, \qquad k = \dots, -1, 0, 1, \dots,$$

forming a system of strips (cylinders), each of which in a section by the plane $x_1 = \text{const}$ gives an $(r-1)$-dimensional cube. Denote by $\gamma_1, \dots, \gamma_T$ the portions of the strips belonging to the domain G and defined in such a way that

$$\bigcup_{k=1}^{T} (G \cap \gamma_k) = G, \qquad \gamma_k \cap \gamma_j = 0 \quad \text{for } k \neq j.$$

Connect all points that have fallen into γ_k by one polygonal line in such a way that, when moving

along it the value x_1 changed monotonically. We connect these polygonal lines by $T-1$ bridges, for example, as follows. First connect into one polygonal line those lying in the strips bounded by the planes $x_s = k_s a n^{-1/r}$ and $x_s = (k_s+1) a n^{-1/r}$, $s = 3, \dots, r$. Then release one coordinate, say x_3, and connect to one another the neighboring connections obtained earlier, and so on. Let us define $s_1(n)$ as the closure of the combined polygonal line.

It is obvious that $s(n) \leqslant s_1(n)$. The result of a direct calculation is

Theorem 1. *Let us choose as a the value at which the following is attained*

$$\inf_{a} \left\{ \alpha^{1-r} \int_{x \in G} \int_{u_2=0}^{1} \cdots \int_{u_r=0}^{1} \int_{t=0}^{\infty} \left[t^2 + \alpha^2 \varphi^2 \sum_{k=2}^{r} u_k^2 \right]^{1/2} \times \frac{f^2}{\varphi^2} \prod_{k=2}^{r} (1-u_k) e^{-tf/\varphi} \, dt \, du_r \dots du_2 \, dx \right\} = M_1(f).$$

Then

$$\lim_{n \to \infty} n^{1/r-1} \mathrm{M} s_1(n) = 2^{r-1} M_1(f), \qquad \lim_{n \to \infty} n^{2/r-1} \mathrm{D} s_1(n) = D_1(f), \qquad \infty > D_1(f) > 0.$$

Here

$$f = f(x) = f(x_1, \dots, x_r), \qquad \varphi = \varphi(x_2, \dots, x_r) = \int f(x_1, \dots, x_r) \, dx_1.$$

It follows from the theorem, in particular, that $M(f) \leqslant 2^{r-1} M_1(f)$. An analogous estimate, contained in (2), has the form $\mu_r \leqslant 6^{-1/2} 12^{1/2r} r^{1/2}$, $r \geqslant 2$, and for $r = 2$ (more strictly) $\mu_2 = 0.920$.

If, for example, $f(x)$ is the density of the uniform distribution in the unit cube, then for $r = 2$ $2M_1(f) = 0.920$, and for $r = 3$ $4M_1(f) = 0.972 < (3/2)^{1/6}$.

In a number of cases the following assertion may also prove useful. Let V be the volume of the domain G.

Theorem 2. *The random variable $s_1(n)$ admits an estimate $s_3(n) \geqslant s_1(n)$ having the properties*

$$\lim_{n\to\infty} n^{1/r-1}\mathrm{M}s_3(n) = \frac{r}{3}(3V)^{1/r},$$

$$\lim_{n\to\infty} n^{2/r-1}\mathrm{D}s_3(n) = \frac{r-1}{15}(3V)^{2/r}.$$

The estimate contained in Theorem 1 can be improved by a more reasonable choice of strips—the width of the strip should in general be variable and, in a neighborhood of the point x, should be $\alpha[nf(x)]^{-1/r}$, where α is a parameter independent of x.

Let us also note that, as $n \to \infty$, we obtain a path equivalent to $s(n)$ (the ratio of the lengths tends to 1) if we unite the shortest paths constructed for the domains $G_1, \dots, G_k$, which form some partition of the domain G (see (1)).

If the integral $\int f^{1-1/r}(x)\,dx$ diverges, then $s(n)$ can grow, generally speaking, arbitrarily fast. This is easy to verify by using the fact that, for large n, with probability close to 1, the value $s(n)$ exceeds the distance from the origin to the most distant point.

An estimate for the number p mentioned at the beginning of the note is not difficult to obtain with the aid of (2) and Theorem 2. For example, for the uniform distribution in the unit square, $p < 0.48$.

2. The assignment problem. Here, for a given matrix

$$A_n = \|a_{ij}\|, \quad i, j = 1, \dots, n,$$

of order n, it is required to indicate a permutation

$$\binom{i}{k_i}, \quad i = 1, \dots, n,$$

for which the sum

$$\sum_{i=1}^{n} a_{ik_i}$$

would be the smallest (or

largest). In other words, one must find the element of the determinant along which the sum of the numbers a_{ij} would be the smallest. We shall assume that the numbers a_{ij} are the result of n^2 independent observations of a random variable ξ with some distribution function $F(x)$. In a number of cases such an assumption may be regarded as justified. Then it proves possible, again by elementary methods, to obtain estimates that hold "almost always," i.e., for "almost all" realizations of the matrices.

Let

$$\xi(n) = \min_{\Pi} \sum_{i=1}^{n} a_{ik_i},$$

where Π is the set of all permutations

$$\begin{pmatrix} i \\ k_i \end{pmatrix}$$

of order n, and let $\xi_n^1 = \xi[A_n]$ be the minimal element of the first column of the matrix A_n. Delete from A_n the first column and the row to which ξ_n^1 belongs. We obtain the truncated matrix $A_{n-1} = A(A_n)$. Denote $\xi_{n-1}^2 = \xi[A_{n-1}]$, $\xi_{n-2}^3 = \xi[A(A_{n-1})]$, and so on. Let, moreover, ξ_n^k be the minimal element of the k-th column of the matrix A_n. Put

$$\xi_1(n) = \xi_n^2 + \xi_{n-1}^2 + \cdots + \xi_1^n, \qquad \xi_2(n) = \xi_n^1 + \cdots + \xi_n^n.$$

Then, evidently,

$$\xi(n) \le \sum_{j=1}^{k} \xi_{n-j+1}^{j} + \xi(n-k),$$

$$\xi_2(n) \le \xi(n) \le \xi_1(n).$$

We shall assume that $\mathbf{M}\xi^2 < \infty$ and that the equation $F(x) = y$ is uniquely solvable in the form $x = F_-(y)$ in a neighborhood of the point $y = 0$. Then for some $\varepsilon > 0$ we shall have

$$\mathbf{M}\left(\xi_n'\right)^k = \int_0^{n\varepsilon} \left(F_-\left(\frac{x}{n}\right)\right)^k \left(1 - \frac{x}{n}\right)^{n-1} dx + O(h^n), \quad h < 1;\ k = 1, 2. \quad (3)$$

It is known ([1]) that the limit theorems describing the distribution of ξ_n^1 for large n differ substantially depending on the properties of $F_-(y)$ in a neighborhood of the point $y = 0$. Let us consider two important special cases.

A. The lower bound of the possible values of ξ is finite. Without loss of generality it may be taken equal to zero. Suppose that, for small x,

$$F(x) = (\alpha x)^\beta (1 + o(1)), \qquad \alpha > 0, \quad \beta > 0, \quad x \ge 0.$$

In this case equalities (3) give, as $n \to \infty$,

$$\mathbf{M}\xi_1(n) \sim \begin{cases} \dfrac{\ln n}{\alpha}, & \text{for } \beta = 1, \\ \dfrac{\Gamma\left(\dfrac{\beta+1}{\beta}\right) n^{1-1/\beta}}{\alpha(1-1/\beta)}, & \text{for } \beta > 1, \\ \text{const}, & \text{for } \beta < 1; \end{cases}$$

$$\mathbf{D}\xi_1(n) \sim \begin{cases} \dfrac{(4-\pi)\ln n}{\alpha^2}, & \text{for } \beta = 2, \\ \dfrac{\Gamma\left(\dfrac{\beta+2}{\beta}\right) - \Gamma^2\left(\dfrac{\beta+1}{\beta}\right)}{\alpha^2(1-2/\beta)} n^{1-2/\beta}, & \text{for } \beta > 2, \\ \text{const}, & \text{for } \beta < 2; \end{cases}$$

$$\mathbf{M}\xi_2(n) \sim \frac{\Gamma\left(\dfrac{\beta+1}{\beta}\right) n^{1-1/\beta}}{\alpha}, \qquad \mathbf{D}\xi_2(n) \sim \frac{\Gamma\left(\dfrac{\beta+2}{\beta}\right) - \Gamma^2\left(\dfrac{\beta+1}{\beta}\right)}{\alpha^2} n^{1-2/\beta}.$$

Hence, in particular, it follows that $\xi(n)/n$ converges in probability to 0, and also that for $\beta > 1$ (for example, the composition of several uniform distributions), with probability 1,

$$\frac{\Gamma\left(\frac{\beta+1}{\beta}\right)}{\alpha} \le \underline{\lim}_{n\to\infty} n^{1/\beta-1}\xi(n) \le \overline{\lim}_{n\to\infty} n^{1/\beta-1}\xi(n) \le \frac{\Gamma\left(\frac{\beta+1}{\beta}\right)}{\alpha(1-1/\beta)}.$$

B. The values of ξ are not bounded from below, and

$$F(x) = \exp(\beta x + \alpha + o(1)) \qquad \text{as } x \to -\infty \quad (\beta > 0).$$

In this case

$$\mathbf{M}\xi_2(n) = -\frac{n}{\beta}(\ln n + \alpha + C + o(1)) \qquad (C \text{ is Euler' s constant}),$$

$$\mathbf{D}\xi_2(n) = \frac{\pi^2}{6\beta^2} n + o(n), \qquad \mathbf{M}\xi_1(n) = \mathbf{M}\xi_2(n) + \frac{n}{\beta} + o(n),$$

$$\mathbf{D}\xi_1(n) = \mathbf{D}\xi_2(n) + o(n),$$

so that $\xi(n)/(n \ln n)$ converges in probability to $-1/\beta$.

If ξ is normally distributed $(0,1)$, then $\xi(n)/(n\sqrt{\ln n}) \to -\sqrt{2}$ (in probability).

There are many ways to improve the estimates obtained.

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Note: Figure translations are in progress. See original paper for figures.