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Abstract

Full Text

HYDROMECHANICS

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ON THE GENERAL THEORY OF STATIONARY MOTIONS IN RELATIVISTIC HYDRODYNAMICS

(Presented by Academician L. I. Sedov on 28 VIII 1961)

Let us consider stationary adiabatic motions of an ideal gas in relativistic hydrodynamics (within the framework of the special theory of relativity). The entropy will be assumed, generally speaking, to be different on different streamlines. In this note it is shown that every such motion reduces to the nonrelativistic motion of a certain gas. For potential plane-parallel motions of an ultrarelativistic gas this was shown in ⁽⁶⁾.

We choose a Galilean coordinate system: $-ds^2 = g_{ik} dx^i dx^k$, $g_{00} = -1$, $g_{11} = g_{22} = g_{33} = 1$, $g_{ik} = 0$ for $i \neq k$, $x^0 = ct$. Latin indices take the values 0, 1, 2, 3, and Greek indices the values 1, 2, and 3.

The basic equations are the laws of conservation of momentum and energy and the law of conservation of the number of particles. We shall take these equations in the form ^(1,4)

$$nu^k \frac{\partial}{\partial x^k} \left(\frac{w}{n} u_i \right) = n \frac{d}{ds} \left(\frac{w}{n} u_i \right) = - \frac{\partial p}{\partial x^i}; \quad (1)$$

$$T \frac{d}{ds} \left(\frac{\sigma}{n} \right) = \frac{d}{ds} \left(\frac{w}{n} \right) - \frac{1}{n} \frac{dp}{ds} = 0, \quad (2)$$

$$\frac{\partial}{\partial x^i} (nu^i) = 0. \quad (3)$$

Here u^i is the 4-velocity, with $u^0 = -u_0 = 1/\sqrt{1-v_\alpha^2/c^2}$, $u^\alpha = u_\alpha = v_\alpha/c\sqrt{1-v_\alpha^2/c^2}$; w, σ, n are respectively the heat function, entropy, and number of particles per unit proper volume; T is the temperature (in the proper reference system of the given fluid element); p is the pressure.

Since stationary motions will be considered, the operator of total differentiation along the world line of a particle, $\frac{d}{ds}$, will have the form

$$u_\alpha \frac{\partial}{\partial x^\alpha}.$$

Introduce the notation

$$wu_\alpha/mcn = \mathbf{v}_\alpha, \quad m^2c^2n^2/w = \tilde{\rho}, \quad w^2/2m^2c^2n^2 = I. \quad (4)$$

In these formulae constants have been introduced: c , the speed of light, and m , the rest mass of a particle, so that the new quantities have respectively the dimensions of velocity, density, and energy per unit mass.

The quantity wu_α/n was first considered by I. M. Khalatnikov ⁽⁴⁾; F. I. Frankl called it the pseudo-velocity ⁽⁵⁾.

With the aid of the notation (4), the original system (1)–(3) in the stationary case is written in the form:

$$\tilde{\rho} \mathbf{v}_\alpha \frac{\partial \mathbf{v}_\beta}{\partial x^\alpha} = -\frac{\partial p}{\partial x^\beta}; \quad (5)$$

$$\tilde{\rho} \mathbf{v}_\alpha \frac{\partial I}{\partial x^\alpha} - \mathbf{v}_\alpha \frac{\partial p}{\partial x^\alpha} = 0, \quad (6)$$

$$\frac{\partial}{\partial x^\alpha} (\tilde{\rho} \mathbf{v}_\alpha) = 0. \quad (7)$$

To these equations one must add the equation of state of the gas

$$F(p, n, w) = 0, \quad (8)$$

which, in the new notation, takes the form

$$\Phi(p, \tilde{\rho}, I) = 0. \quad (9)$$

Equations (5)–(7), (9) may be regarded as nonrelativistic equations for stationary adiabatic motions of a nonrelativistic gas with equation of state (9), where \mathbf{v} is its velocity, $\tilde{\rho}$ its density, p the pressure, and I the heat function per unit mass. Unlike the original relativistic gas, we shall call this gas auxiliary. In the nonrelativistic limit the two gases coincide. In this connection $c^2/2$ must be subtracted from I .

From the thermodynamic identity for a relativistic gas $d(w/n) = T d(\sigma/n) + dp/n$, which in the notation (4) takes the form $dI = d(\sigma/n)Tw/m^2c^2n + dp/\tilde{\rho}$, it follows that the auxiliary gas must be assigned a temperature \tilde{T} and an entropy per unit mass \tilde{S} equal to

$$\tilde{T} = Tw/mc^2n, \quad \tilde{S} = \sigma/mn. \quad (10)$$

A perfect Boltzmann relativistic gas ⁽²⁾ corresponds to an auxiliary gas satisfying Clapeyron's equation, but with a complicated dependence $I(\tilde{T})$ ⁽²⁾.

The ultrarelativistic case for such a gas ⁽³⁾ corresponds to a perfect auxiliary gas with constant heat capacities, whose ratio is $\varkappa = c_p/c_v = 2$.

We shall denote the speed of sound in the relativistic gas by ω , and the speed of sound in the auxiliary gas by a . Taking into account that $\omega^2 w/c^2 n^2 = (d(w/n)/dn)_{\sigma/n=\text{const}}$, we find the relation between a^2 and ω^2 :

$$a^2 = \left(\frac{\partial p}{\partial \tilde{\rho}} \right)_{\tilde{S}=\text{const}} = \frac{1}{m^2 c^2} \left(\frac{\partial p}{\partial (n^2/w)} \right)_{\sigma/n=\text{const}} = \frac{w^2 \omega^2}{m^2 c^2 n^2 (c^2 - \omega^2)}. \quad (11)$$

Thus, the speeds of sound a and ω do not coincide.

The Mach number in the relativistic gas $M = \sqrt{\mathbf{v}_\alpha^2}/\omega$ and the Mach number in the auxiliary gas $\tilde{M} = \sqrt{\mathbf{v}_\alpha^2}/a$ are related by $\tilde{M}^2(1 - M^2\omega^2/c^2) = M^2(1 - \omega^2/c^2)$, from which it follows that they simultaneously become equal to unity and that for $M < 1$ we have $\tilde{M} < 1$, and conversely.

The system of streamlines in the two gases is the same. From the Bernoulli equation $\mathbf{v}_\alpha^2/2 + I = \text{const}$ along a streamline of the auxiliary gas, there follows immediately the conservation, along a streamline in the relativistic gas, of the quantity wu^0/n (the relativistic Bernoulli equation (1)).

For potential motions wu^0/n is constant throughout the flow; in this case \mathbf{v}_α proves to be proportional to v_α , and $\tilde{\rho}$ has the meaning of the density of the number of particles of the original gas in the laboratory system.

In an ultrarelativistic gas, when the number of particles is not conserved and the chemical potential $\mu = 0$, the fundamental equations should be taken in the form ⁽⁴⁾

$$u^\alpha \frac{\partial}{\partial x^\alpha} (Tu_\beta) = -\frac{\partial T}{\partial x^\beta}, \quad \frac{\partial}{\partial x^\alpha} (\sigma u^\alpha) = 0, \quad \sigma = k_0 T^3, \quad k_0 = \text{const}.$$

Introducing the notation $\vartheta_\alpha = k_1 T u_\alpha$, $\tilde{\rho} = k_2 T^2$; k_1 and k_2 are dimensional constants, we also obtain the nonrelativistic equations for isentropic motions of a perfect gas with heat-capacity ratio $\varkappa = c_p/c_v = 2$ ($p = k_1^2 \tilde{\rho}^2 / 4k_2$).

In this case, for potential motions the Bernoulli equation has the form $T/\sqrt{1 - v_\alpha^2/c^2} = \text{const}$ throughout the flow; the velocity ϑ_α is proportional to the original velocity v_α , and $\tilde{\rho}$ corresponds to the temperature in the laboratory system.

Thus, the results of the nonrelativistic theory of stationary motions, in particular of one-dimensional, plane-parallel, and axisymmetric flows, carry over to relativistic gas dynamics.

It is interesting to note that these results make it possible to obtain the conditions at a shock wave ^(1,7) and to formulate Zemplén's theorem ⁽⁸⁾ in a relativistic gas by means of a simple recalculation from the nonrelativistic relations. This is connected with the fact that the conservation laws for the fluxes of momentum, energy, and particle number are obtained from that part of the basic equations which contains only spatial derivatives, i.e., from the system (5)–(7), (9), which simultaneously describes a nonrelativistic gas. Here the coordinate system is understood to be attached to the wave. The entire nonrelativistic thermodynamic analysis of the shock adiabat ⁽¹⁾ proves valid, in the variables $\vartheta, \tilde{\rho}, p, I, \tilde{T}, \tilde{S}$, also for the relativistic gas. Relations in the original variables are obtained after substitution according to formulas (4), (10), and (11). In particular, the fundamental thermodynamic inequality

$$(\partial^2 p / \partial (1/\tilde{\rho})^2)_{\tilde{S}=\text{const}} > 0,$$

which leads to the existence specifically of compression shocks, is written in the original variables in the form

$$(\partial^2 p / \partial (w/n^2)^2)_{\sigma/n=\text{const}} > 0.$$

In these last arguments, use is made in an essential way of the fulfillment of the particle conservation law (3).

We also emphasize that the discussion concerns shock waves that may propagate in an arbitrary manner.

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