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Abstract

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ON THE NOMOGRAPHING OF SOLUTIONS OF A SECOND-ORDER DIFFERENTIAL EQUATION

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The second-order differential equation

$$\frac{d^2v}{du^2} = F\left(u, v, \frac{dv}{du}\right) \tag{1}$$

will be called nomographable under the boundary conditions

$$u = u_1, \quad v = v_1 \quad \text{and} \quad u = u_2, \quad v = v_2, \tag{2}$$

if there exists a Massau determinant

$$\begin{vmatrix} A_1(u) & A_2(u) & A_3(u) \\ B_1(v) & B_2(v) & B_3(v) \\ c_1 & c_2 & 1 \end{vmatrix} \not\equiv 0, \tag{3}$$

satisfying the condition

$$\begin{vmatrix} A_1(u) & A_2(u) & A_3(u) \\ B_1(\bar{v}) & B_2(\bar{v}) & B_3(\bar{v}) \\ c_1 & c_2 & 1 \end{vmatrix} \equiv 0, \tag{4}$$

where $\bar{v} = v(u, c_1, c_2)$ is the general solution of equation (1), and c_1, c_2 are arbitrary constants. In this case the point M with mark u of the scale u, the point N with mark v of the scale v, and the point C with coordinates $(c_1; c_2)$ lie on one straight line. The point $C(c_1; c_2)$ is determined, by means of the boundary conditions (2), as the point of intersection of the straight lines M_1N_1 and M_2N_2, passing respectively through the points $M_1(u_1), N_1(v_1)$ and $M_2(u_2), N_2(v_2)$ (see Fig. 1).

Fig. 1

Fig. 1

Figure 1: Fig. 1

Theorem. *A necessary and sufficient condition for the nomographability of equation (1) is that its general solution be a solution of the differential equation of geodesic lines on a sphere under the conditions*

$$\left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\} = 0, \qquad \left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\} = 0. \tag{5}$$

1. Suppose that equation (1) is nomographable. Then it follows from (4) that its general solution has the form

$$c_1 P_{23}(u,v) + c_2 P_{31}(u,v) + P_{12}(u,v) = 0, \tag{6}$$

where

$$P_{ik} = \begin{vmatrix} A_i & A_k \\ B_i & B_k \end{vmatrix}. \tag{7}$$

It is known that the vector $\mathbf{P} = \{P_{23}; P_{31}; P_{12}\}$ satisfies conditions (2)

$$(\mathbf{P}\mathbf{P}'_u\mathbf{P}''_{uu}) \equiv 0, \tag{8}$$

$$(\mathbf{P}\mathbf{P}'_v\mathbf{P}''_{vv}) = 0, \tag{9}$$

$$(\mathbf{P}\mathbf{P}'_u\mathbf{P}''_v) \neq 0. \tag{10}$$

Conditions (8), (9), (10) are necessary and sufficient for the left-hand side of equation (6) to be representable in the form of the Massau determinant (3).

From (6) we obtain

$$\begin{aligned} c_1 q_{23} + c_2 q_{31} + q_{12} = 0, \\ c_1 r_{23} + c_2 r_{31} + r_{12} = 0, \end{aligned} \tag{11}$$

where

$$q_{ik} = \frac{\partial P_{ik}}{\partial u} + \frac{\partial P_{ik}}{\partial v}\frac{dv}{du},$$

$$r_{ik} = \frac{\partial^2 P_{ik}}{\partial u^2} + 2\frac{\partial^2 P_{ik}}{\partial u\,\partial v}\frac{dv}{du} + \frac{\partial^2 P_{ik}}{\partial v^2}\left(\frac{dv}{du}\right)^2 + \frac{\partial P_{ik}}{\partial v}\frac{d^2 v}{du^2}. \tag{12}$$

Eliminating c_1, c_2 from (6), (11), we obtain the desired differential equation

$$\frac{d^2v}{du^2}(\mathbf{P}\mathbf{P}'_u\mathbf{P}'_v) + \left(\frac{dv}{du}\right)^3(\mathbf{P}\mathbf{P}'_v\mathbf{P}''_{vv}) + \left(\frac{dv}{du}\right)^2[(\mathbf{P}\mathbf{P}'_u\mathbf{P}''_{vv}) + 2(\mathbf{P}\mathbf{P}'_v\mathbf{P}''_{uv})] +$$

$$+\left(\frac{dv}{du}\right)[(\mathbf{P}\mathbf{P}'_v\mathbf{P}''_{uu}) + 2(\mathbf{P}\mathbf{P}'_u\mathbf{P}''_{uv})] + (\mathbf{P}\mathbf{P}'_u\mathbf{P}''_{uu}) = 0. \tag{13}$$

Consider the functions $x = \lambda(u,v)P_{23},\ y = \lambda(u,v)P_{31},\ z = \lambda(u,v)P_{12}$. Choose $\lambda(u,v)$ so that $x^2+y^2+z^2=1$. Then x, y, z are the coordinates of a point of the sphere whose radius is equal to unity. If, in equations (8), (9), (10), (13), P_{23}, P_{31}, P_{12} are replaced respectively by $x/\lambda,\ y/\lambda,\ z/\lambda$, then these equations will have the form:

$$(\vec\rho\,\vec\rho'_u\vec\rho''_{uu}) \equiv 0, \tag{8'}$$

$$(\vec\rho\,\vec\rho'_v\vec\rho''_{vv}) \equiv 0, \tag{9'}$$

$$(\vec\rho\,\vec\rho'_u\vec\rho'_v) \neq 0, \tag{10'}$$

$$(\vec\rho\,\vec\rho'_u\vec\rho'_v)\frac{d^2v}{du^2} + \left(\frac{dv}{du}\right)^3(\vec\rho\,\vec\rho'_v\vec\rho''_{vv}) + \left(\frac{dv}{du}\right)^2[(\vec\rho\,\vec\rho'_u\vec\rho''_{vv}) + 2(\vec\rho\,\vec\rho'_v\vec\rho''_{uv})] +$$

$$+\frac{dv}{du}[(\vec\rho\,\vec\rho'_v\vec\rho''_{uu}) + 2(\vec\rho\,\vec\rho'_u\vec\rho''_{uv})] + (\vec\rho\,\vec\rho'_u\vec\rho''_{uu}) = 0, \tag{13'}$$

where $\vec\rho = \{x; y; z\}$.

From identities $(8')$, $(9')$ we obtain the Gauss derivative formulas

$$\vec\rho''_{uu} = \left\{ {11 \atop 1} \right\}\vec\rho'_u + D\vec\rho, \qquad \vec\rho''_{vv} = \left\{ {22 \atop 2} \right\}\vec\rho'_v + D''\vec\rho. \tag{14}$$

Moreover, it is known that

$$\vec\rho''_{uv} = \left\{ {12 \atop 1} \right\}\vec\rho'_u + \left\{ {12 \atop 2} \right\}\vec\rho'_v + D'\vec\rho. \tag{14'}$$

If in equation (13′) we replace $\vec{\rho}''_{uu}, \vec{\rho}''_{uv}, \vec{\rho}''_{vv}$ by their values from formulas (14), (14′), then it takes the form

$$\frac{d^2v}{du^2} + \left(\begin{Bmatrix}22\\2\end{Bmatrix} - 2\begin{Bmatrix}12\\1\end{Bmatrix}\right)\left(\frac{dv}{du}\right)^2 + \left(-\begin{Bmatrix}11\\1\end{Bmatrix} + 2\begin{Bmatrix}12\\2\end{Bmatrix}\right)\left(\frac{dv}{du}\right). \tag{15}$$

This is the differential equation of geodesic lines on the sphere under the conditions

$$\begin{Bmatrix}22\\1\end{Bmatrix} = 0, \qquad \begin{Bmatrix}11\\2\end{Bmatrix} = 0. \tag{16}$$

Thus, *if equation* (1) *is nomographable, then its general solution is the solution of the differential equation of geodesics on the sphere under conditions* (16).

2. Let us prove the converse proposition to the one considered, i.e., that *the solution of equation* (15) *under conditions* (16) *is nomographable.*

The general solution of the differential equation of geodesics on the sphere has the form

$$c_1x(u,v) + c_2y(u,v) + z(u,v) = 0, \tag{17}$$

where x, y, z are the coordinates of a point of the sphere. Since in equation (15)

$$\begin{Bmatrix}11\\2\end{Bmatrix} = 0, \qquad \begin{Bmatrix}22\\1\end{Bmatrix} = 0,$$

the equalities (14) hold. Consequently,

$$(\vec{\rho}\,\vec{\rho}'_u\,\vec{\rho}''_{uu}) \equiv 0, \qquad (\vec{\rho}\,\vec{\rho}'_v\,\vec{\rho}''_{vv}) \equiv 0.$$

The vector $\vec{\rho}$ is the radius vector of a point of the sphere and at the same time the unit normal vector to the sphere; therefore

$$(\vec{\rho}\,\vec{\rho}'_u\,\vec{\rho}'_v) \not\equiv 0.$$

On the basis of conditions (8′), (9′), (10′), expression (17) can be represented in the form of a Massau determinant; consequently, equation (15) is nomographable as (1).

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References Cited

[1] S. V. Bakhvalov, *Vestn. Mosk. Univ.*, No. 1 (1961). [2] O. D. Kellogg, *Zs. f. Math. u. Phys.* (1914).

Note: Figure translations are in progress. See original paper for figures.