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Let us consider the differential equation

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Abstract

Full Text

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EXPANSION IN EIGENFUNCTIONS OF NON-SELF-ADJOINT SINGULAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

(Presented by Academician A. N. Kolmogorov on 12 I 1962)

Let us consider the differential equation

$$-y'' + q(x)y = \lambda^2 y \quad (-\infty < x < \infty), \tag{1}$$

where $q(x)$ is an arbitrary complex-valued function summable on each finite interval of the axis $-\infty < x < \infty$, and let us denote by $\omega_1(\lambda, x)$, $\omega_2(\lambda, x)$ the solution of this differential equation satisfying the conditions

$$\omega_1(\lambda, 0) = 1, \quad \omega_1'(\lambda, 0) = 0;$$

$$\omega_2(\lambda, 0) = 0, \quad \omega_2'(\lambda, 0) = 1. \tag{2}$$

In the work of V. A. Marchenko ([3]), equation (1) was considered on the half-axis $[0, \infty)$ with the boundary condition

$$y'(0) - hy(0) = 0, \tag{3}$$

where h is an arbitrary complex number. The following result was obtained:

Theorem. Denote by Z the linear topological space of even entire functions of exponential type, summable on the real axis; by $T(Z)$ the space conjugate to it. Then to every boundary-value problem (1)–(3) there corresponds a certain generalized function $R \in T(Z)$ such that

$$\int_0^\infty f(x)g(x)\,dx = (R,\ E_f(\lambda)E_g(\lambda)),$$

where $f(x)$ and $g(x)$ are arbitrary finite functions belonging to $L^2(0, \infty)$,

$$E_f(\lambda) = \int_0^\infty f(x)\omega(\lambda, x)\,dx,$$

($\omega(\lambda, x)$ is the solution of equation (1) satisfying condition (3)).

Further, in the work ([3]) necessary and sufficient conditions for the existence of a spectral function $R \in T(Z)$ were established, and it was proved that to every function $R \in T(Z)$ satisfying these conditions there corresponds a certain boundary-value problem (1)–(3).

In the present paper the results obtained by V. A. Marchenko are transferred to equation (1), given on the whole interval $(-\infty, \infty)$. The method used in the work is the one first applied by V. A. Marchenko.

Denote by K the space of infinitely differentiable finite functions $f(x)$. Consider the functions

$$C_f(\lambda) = \int_{-\infty}^\infty f(x)\cos\lambda x\,dx; \qquad S_f(\lambda) = \int_{-\infty}^\infty f(x)\frac{\sin\lambda x}{\lambda}\,dx.$$

As is known (see (2), p. 194), the functions $C_f(\lambda)$ form a topological space Z of even entire functions of finite degree $\psi(\lambda)$, satisfying the inequalities

$$|\lambda|^q|\psi(\lambda)| \le C_q e^{a|\tau|}, \qquad q = 0, 1, \dots, \qquad \tau = \operatorname{Im}\lambda, \qquad 0 < a < \infty.$$

It is easy to show that the functions $S_f(\lambda)$ also belong to Z. We define generalized functions on Z as linear continuous functionals $R[F(\lambda)]$, putting

$$R[F(\lambda)] = (R, F(\lambda)).$$

The totality of all generalized functions defined in this way will be denoted by $T(Z)$. A sequence $R_n \in T(Z)$ converges to $R \in T(Z)$ if

$$\lim_{n\to\infty}(R_n, F(\lambda)) = (R, F(\lambda))$$

for all basic functions $F(\lambda) \in Z$.

Let $A(x)$ be an arbitrary locally summable function. Put

$$A_n(x) = \begin{cases} A(x), & |x| \le n, \\ 0, & |x| > n, \end{cases}$$

and denote

$$C_{A_n}(\lambda) = \int_{-\infty}^{\infty} A_n(x) \cos \lambda x \, dx, \qquad S_{A_n}(\lambda) = \int_{-\infty}^{\infty} A_n(x) \frac{\sin \lambda x}{\lambda} \, dx.$$

We define the C- and S-Fourier transforms of the function $A(x)$ by putting

$$C_A = \lim_{n \to \infty} C_{A_n}, \qquad S_A = \lim_{n \to \infty} S_{A_n},$$

where convergence is understood in the sense of generalized functions.

A fundamental role in proving the main results is played by

Theorem (see (1)). The solutions $\omega_1(\lambda, x)$, $\omega_2(\lambda, x)$ of equation (1) can be expressed through $\cos \lambda x$ and $\frac{\sin \lambda x}{\lambda}$, respectively, with the aid of transformation operators in the form

$$\omega_1(\lambda, x) = \cos \lambda x + \int_{-|x|}^{|x|} K(x,t) \cos \lambda t \, dt,$$

$$\omega_2(\lambda, x) = \frac{\sin \lambda x}{\lambda} + \int_{-|x|}^{|x|} K(x,t) \frac{\sin \lambda x}{\lambda} \, dt; \tag{4}$$

$$\cos \lambda x = \omega_1(\lambda, x) - \int_{-|x|}^{|x|} H(x,t) \omega_1(\lambda, t) \, dt,$$

$$\frac{\sin \lambda x}{\lambda} = \omega_2(\lambda, x) - \int_{-|x|}^{|x|} H(x,t) \omega_2(\lambda, t) \, dt, \tag{5}$$

where the kernels $K(x,t)$ and $H(x,t)$ are absolutely continuous in both variables.

We define the ω-Fourier transforms of a function $f(x) \in K$ by the equalities

$$E_f(\lambda) = \int_{-\infty}^{\infty} f(x) \omega_1(\lambda, x) \, dx, \qquad G_f(\lambda) = \int_{-\infty}^{\infty} f(x) \omega_2(\lambda, x) \, dx.$$

With the aid of the preceding theorem it is easy to show that the set of ω-Fourier transforms of all functions from K coincides with Z. We can now formulate the principal results obtained; we shall not dwell on their proof.

Theorem. To each differential equation (1) with solutions satisfying condition (2) there corresponds a spectral matrix-function of the second order $\|R_{ik}\|_{i,k=1,2}$, $R_{ik} \in T(Z)$, such that

$$\int_{-\infty}^{\infty} f(x)g(x)\,dx = (R_{11}, E_f(\lambda)E_g(\lambda)) + (R_{12}, E_f(\lambda)G_g(\lambda)) +$$

$$+(R_{21}, G_f(\lambda)E_g(\lambda)) + (R_{22}, G_f(\lambda)G_g(\lambda)),$$

where $f(x)$, $g(x) \in K$; $E_f(\lambda)$, $G_f(\lambda)$, $E_g(\lambda)$, $G_g(\lambda)$ are their ω-Fourier transforms. The elements of the spectral matrix-function R_{ik} are connected with the kernel $H(x,t)$ of the transformation operator (5) by the formulas

$$R_{11} = \frac{1}{2\pi}(1 - C_H), \qquad R_{12} = R_{21} = -\frac{\lambda^2}{2\pi}S_H, \qquad R_{22} = \frac{\lambda^2}{2\pi}(1 + C_H),$$

where C_H and S_H are the C- and S-Fourier transforms of the locally summable function $H(x,0)$.

Corollary. If the function $f(x) \in K$, then the formula

$$f(x) = (R_{11}, E_f(\lambda)\omega_1(\lambda,x)) + (R_{12}, E_f(\lambda)\omega_2(\lambda,x)) +$$

$$+(R_{21}, G_f(\lambda)\omega_1(\lambda,y)) + (R_{22}, G_f(\lambda)\omega_2(\lambda,x))$$

holds.

As is known ([2], p. 180), for every finite continuous function $\theta(t)$ there exists a sequence of functions $\theta_\varepsilon(t) \in K$ with the same interval of finiteness $[-a,a]$ such that

$$\lim_{\varepsilon\to 0} \theta_\varepsilon(t) = \theta(t)$$

uniformly in t. Let $F(\lambda)$ and $F_\varepsilon(\lambda)$ be their Fourier transforms. Then it is easy to show that

$$\lim_{\varepsilon\to 0} F_\varepsilon(\lambda) = F(\lambda)$$

uniformly in λ. We extend the functional $R \in T(Z)$ to the set of all such functions $F(\lambda)$, putting

$$(R, F(\lambda)) = \lim_{\varepsilon\to 0}(R, F_\varepsilon(\lambda)).$$

Theorem. In order that a generalized matrix-function

$$\|R_{ik}\|_{i,k=1,2}, \qquad R_{ik} \in T(Z), \qquad R_{12} = R_{21}, \qquad R_{22} = \frac{\lambda^2}{\pi}(1 - \pi R_{11})$$

be the spectral matrix-function of some problem (1)–(2), it is necessary and sufficient that the following conditions be satisfied:

1) In Z there do not exist nonzero functions $E_f(\lambda)$, $G_f(\lambda)$ satisfying the equality

$$(R_{11}, E_f(\lambda)E_y(\lambda)) + (R_{12}, E_f(\lambda)G_y(\lambda)) +$$

$$+(R_{21}, G_f(\lambda)E_y(\lambda)) + (R_{22}, G_f(\lambda)G_y(\lambda)) = 0$$

for all $E_y(\lambda)$, $G_y(\lambda) \in Z$.

2) The functions

$$\Phi_1(x) = \left(R_{11},\, 2\frac{1-\cos\lambda x}{\lambda^2}\right) \qquad (-\infty < \lambda < \infty);$$

$$\Phi_2(x) = \left(R_{21},\, -\frac{2x\sin\lambda x}{\lambda^3} - \frac{4(1-\cos\lambda x)}{\lambda^4}\right) \qquad (-\infty < \lambda < \infty)$$

have two absolutely continuous derivatives and

$$\lim_{x\to 0}\frac{\Phi_2''(x)}{x} = 0, \qquad \Phi_1''(0) = 0.$$

Here the function $q(x)$ in equation (1) has as many absolutely continuous derivatives as $\Phi_1''(x)$ and $\Phi_2''(x)$ have, provided that

$$\lim_{x\to 0}\frac{\Phi_2^{(k+2)}(x)}{x} \neq \infty, \quad k = 1, 2, \ldots .$$

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Note: Figure translations are in progress. See original paper for figures.