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Abstract

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MATHEMATICS

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ON EXTREMAL PROBLEMS FOR ORLICZ CLASSES OF ANALYTIC FUNCTIONS IN THE UNIT DISK

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§ 1. Numerous works have been devoted to the question of studying extremal problems for analytic functions of the classes H_p (for the definition of these classes see ([1])) (see, for example, ([2–7]), where the preceding literature is also indicated). In the present paper extremal problems are investigated for classes of analytic functions with the Orlicz metric. The known results for the classes H_p are special cases of the theorems given below.

We shall use the notation and definitions of the book ([8]). In particular, we shall denote by $p(t)$ and $q(s)$ the right inverse functions;

$$M(u) = \int_0^{|u|} p(t)\,dt \quad \text{and} \quad N(v) = \int_0^{v} q(s)\,ds$$

the mutually complementary N-functions; L_M^* the Orlicz space; E_M^* the closure in L_M^* of the set of bounded functions; $\|f\|_M$ the Orlicz norm of an element $f \in L_M^*$, and $\|f\|_{(M)}$ the Luxemburg norm.

First we define the classes in which the extremal problems will be solved. Let $f(z)$ be analytic in the disk $|z| < 1$. Put

$$\|f\|_{(M)}^r = \inf k \tag{1}$$

over all $k > 0$ for which

$$\int_0^{2\pi} M\left[\frac{|f(re^{i\theta})|}{k}\right] d\theta \le 1,$$

and

$$\|f\|_M^r = \sup \int_0^{2\pi} |f(re^{i\theta})\,v(e^{i\theta})|\,d\theta, \qquad \int_0^{2\pi} N[v(e^{i\theta})]\,d\theta \le 1. \tag{2}$$

It is easy to show that $\|f\|^r_{(M)}$ and $\|f\|^r_M$ are increasing functions of r. Denote by H^*_M the class of functions analytic in $|z| < 1$ for which

$$\sup_r \|f\|^r_M < +\infty.$$

For $f \in H^*_M$ put

$$\|f(z)\|_M = \|f\|_M = \sup_r \|f\|^r_M = \lim_{r\to1} \|f\|^r_M; \tag{3}$$

$$\|f(z)\|_{(M)} = \|f\|_{(M)} = \sup_r \|f\|^r_{(M)} = \lim_{r\to1} \|f\|^r_{(M)}. \tag{4}$$

If in H^*_M one introduces the norm defined by equality (3) or by equality (4), then H^*_M becomes a Banach space; moreover, the norms $\|f\|_M$ (the Orlicz norm) and $\|f\|_{(M)}$ (the Luxemburg norm) are equivalent to each other. Obviously, H^*_M is contained in H_1; therefore every function $f(z) \in H^*_M$ has angular boundary values almost everywhere. In what follows we shall sometimes identify a function $f(z) \in H^*_M$ with its angular boundary ···

values $f(e^{i\theta})$. Under this condition, with the aid of the results of paper ([9]), the following theorem can be obtained:

Theorem 1. $H^*_M = H_1 \cap L^*_M$, and in this case

$$\|f(e^{i\theta})\|_M = \|f(z)\|_M = \|f\|_M,$$

$$\|f(e^{i\theta})\|_{(M)} = \|f(z)\|_{(M)} = \|f\|_{(M)},$$

where $\|f(e^{i\theta})\|_M(\|f(e^{i\theta})\|_{(M)})$ denotes the Orlicz (Luxemburg) norm of the boundary values $f(e^{i\theta})$ of the function $f(z)$.

By EH_M we shall denote the closure in the norm in H^*_M of the set of bounded analytic functions.

Theorem 2. $EH_M = H_1 \cap E_M$.

Denote by H^{*1}_M the unit sphere in the space H^*_M with norm (3), and by $H^{*1}_{(M)}$ the unit sphere in the same space with norm (4).

Theorem 3. H^{*1}_M, $H^{*1}_{(M)}$ are compact in themselves in the sense of convergence inside $|z| < 1$.

Theorem 4. In order that a sequence $\{f_n\} \in H^*_M$ converge E_N-weakly, it is necessary and sufficient that the norms $\|f_n\|_M$ be uniformly bounded and the sequence $\{f_n(z)\}$ converge uniformly inside the disk $|z| < 1$.

§ 2. We now pass to the main question: the study of extremal problems in our classes.

Theorem 5. For any $v(e^{i\theta}) \in L_N^*$ we have

$$\sup_{\substack{f\in EH_M \\ \|f\|_{(M)}\le 1}} \left| \int_0^{2\pi} f(e^{i\theta})\, v(e^{i\theta})\, d\theta \right| = \sup_{f\in H^{*1}_{(M)}} \left| \int_0^{2\pi} f(e^{i\theta})\, v(e^{i\theta})\, d\theta \right| = \inf_{\varphi\in H_N^*} \|v-\varphi\|_N, \tag{5}$$

and there exists an extremal function $\varphi^*(e^{i\theta}) \in H_N^*$.

Theorem 6. If $v(e^{i\theta}) \in E_N$, then there exists an extremal function in the left-hand side of equality (5). If $q(s)$ is a continuous function, then the extremal functions $f^*(e^{i\theta})$ and $\varphi^*(e^{i\theta})$ are connected by the relations:

$$f^*(e^{i\theta})\,[v(e^{i\theta}) - \varphi^*(e^{i\theta})]\, e^{i\theta} = e^{i\alpha} q[k^*|v(e^{i\theta}) - \varphi^*(e^{i\theta})|]\ |v(e^{i\theta}) - \varphi^*(e^{i\theta})| \tag{6}$$

or

$$f^*(e^{i\theta})[v(e^{i\theta}) - \varphi^*(e^{i\theta})]e^{i\theta} = e^{i\alpha}\frac{1}{k^*}\,|f^*(e^{i\theta})|\, p[|f^*(e^{i\theta})|], \tag{7}$$

where also $\|f^*\|_{(M)} = 1$ and, moreover,

$$\int_0^{2\pi} M[|f^*(e^{i\theta})|]\, d\theta = 1.$$

Here α is a real number, and $k^* > 0$ satisfies the condition

$$\|v-\varphi^*\|_N = \frac{1}{k^*}\left\{1 + \int_0^{2\pi} N[k^*|v(e^{i\theta}) - \varphi^*(e^{i\theta})|]\ d\theta\right\}.$$

(The existence of such a k^* for continuous $q(s)$ follows from ([8]), pp. 106–110.)

Conversely, if for two functions $f^*(e^{i\theta}) \in H_M^*$ and $\varphi^*(e^{i\theta}) \in H_N^*$ the relations (6) or (7) hold and

$$\int_0^{2\pi} M[|f^*(e^{i\theta})|]\, d\theta = 1,$$

then $f^*(e^{i\theta})$ and $\varphi^*(e^{i\theta})$ are extremal functions in equality (5).

The extremal function $\varphi^*(z)$ is always unique, while the extremal function $f^*(z)$ is unique up to a factor $e^{i\alpha}$.

Theorem 7. If $v(z)$ is a rational function and $\beta_1, \dots, \beta_n$ are all its poles inside $|z| < 1$, then

$$\Phi(z) = zf^*(z)[v(z) - \varphi^*(z)] = \frac{A}{K^*}\, z\, \frac{\prod_1^{n-1}(z-\alpha_i)(1-\overline{\alpha_i}z)}{\prod_1^{n}(z-\beta_i)(1-\overline{\beta_i}z)}; \tag{8}$$

$$f^*(z)=\prod{}' \frac{z-\alpha_i}{1-\overline{\alpha_i}z}\exp\left[\frac{1}{2\pi}\int_0^{2\pi}\ln\left\{S^{-1}\left[|A|\frac{\prod_1^{n-1}|e^{i\theta}-\alpha_i|^2}{\prod_1^n|e^{i\theta}-\beta_i|^2}\right]\right\}\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\theta\right]; \tag{9}$$

$$v(z)-\varphi^*(z)=\frac{A}{K^*}\prod{}'' \frac{z-\alpha_i}{1-\overline{\alpha_i}z}\prod_1^{n-1}(1-\overline{\alpha_i}z)^2\prod_1^n\left[(z-\beta_i)(1-\overline{\beta_i}z)\right]^{-1}\times$$

$$\times\exp\left[\frac{1}{2\pi}\int_0^{2\pi}\ln\left\{S^{-1}\left[|A|\frac{\prod_1^{n-1}|e^{i\theta}-\alpha_i|^2}{\prod_1^n|e^{i\theta}-\beta_i|^2}\right]\right\}\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\theta\right], \tag{10}$$

where α_i ($|\alpha_i|<1$), $i=1,\dots,n-1$, are certain points; $\prod'$ extends over those α_i which serve as zeros of $f^*(z)$, and $\prod''$ over those α_i which serve as zeros of $v(z)-\varphi^*(z)$; $S^{-1}(\omega)$ is the function inverse to the function $s(\omega)=\omega p(\omega)$ ($\omega\ge 0$); A satisfies the condition

$$\int_0^{2\pi}M\left\{S^{-1}\left[|A|\frac{\prod_1^{n-1}|e^{i\theta}-\alpha_i|^2}{\prod_1^n|e^{i\theta}-\beta_i|^2}\right]\right\}d\theta=1. \tag{11}$$

From formulas (8), (9) one immediately obtains the form of the extremal functions for the classes H_p ($p\ge 1$) and for the class of bounded functions (see (2–7)). Indeed, for H_p, $p\ge 1$, we have $S(\omega)=\omega^p$, and formula (9) gives, for example,

$$f^*(z)=C\prod{}' \frac{z-\alpha_i}{1-\overline{\alpha_i}z}\prod_1^{n-1}(1-\overline{\alpha_i}z)\cdot\prod_1^n(1-\overline{\beta_i}z)^{-2/p}.$$

§ 3. Applying the methods developed in the works ([10,11]), one can investigate, by means of Theorem 7, the form of the extremal functions in problems of Carathéodory–Fejér–Nevanlinna–Pick type for our classes of functions.

Theorem 8. Let linear functionals $l_1(f),\dots,l_m(f)$ of the form

$$l_i(f)=\int_0^{2\pi}f(e^{i\theta})\,v_i(e^{i\theta})\,d\theta,\qquad v_i(e^{i\theta})\in E_N,\quad i=1,\dots,m,$$

and arbitrary complex numbers $c_1,\dots,c_m$ be given. Among the functions of the class H_M^* for which $l_i(f)=c_i$, $i=1,\dots,m$, the smallest norm $\|f\|_{(M)}$ is attained by a unique function $f^*(z)$, such that $f(z)=f^*(z)\|f\|_{(M)}$ will be extremal for some functional $l(f)$ which is a linear combination of the functionals

$l_1(f), \dots, l_m(f)$. If $v_1(z), \dots, v_m(z)$ are rational, then the function $f^*(z)$ has the form

$$f^*(z) = \|f^*\|_{(M)} \prod\nolimits' \frac{z-\alpha_i}{1-\overline{\alpha_i}z} \exp\left[\frac{1}{2\pi}\int_0^{2\pi} \ln\left\{S^{-1}\left[|A|\frac{\prod_1^{n-1}|e^{i\theta}-\alpha_i|^2}{\prod_1^n |e^{i\theta}-\beta_i|^2}\right]\right\}\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\theta\right], \tag{12}$$

where $\beta_1, \dots, \beta_n$ are all the poles of the functions $v_1, \dots, v_m$; $\alpha_1, \dots, \alpha_{n-1}$ are certain points in the disk $|z| < 1$; $\prod'$ extends over some of the α_i; A satisfies condition (11).

Theorem 9. For any $f(z) \in H_M^*$ there exists a sequence $\{\varphi_n(z)\} \in H_M^*$ such that $\{\varphi_n(z)\}$ converges uniformly inside $|z| < 1$ to $f(z)$, and moreover

$$\|\varphi_n\|_{(M)} = \|f\|_{(M)} \tag{13}$$

and the functions $\varphi_n(z)$ have the form (9).

Denote by $H_{(M)}^{*1}(a)$ the set consisting of those functions $f(z) \in H_{(M)}^{*1}$ for which $\|f\|_{(M)} \leqslant 1$, $f(0) = 0$, and $f'(0) = a \neq 0$. If $|a| > M^{-1}(1/2\pi)$, then this set is empty. For $|a| = M^{-1}(1/2\pi)$ this set consists of the single function $f(z) = az$.

For $|a| < M^{-1}(1/2\pi)$ there exists a disk $|z| < r < 1$ such that all functions of the class $H_{(M)}^{*1}(a)$ are univalent in it, while for any R, $R > r$, there are functions from $H_{(M)}^{*1}(a)$ that are not univalent in the disk $|z| < R$. The number r will be called the radius of univalence of the class $H_{(M)}^{*1}(a)$. A function $f^*(z) \in H_{(M)}^{*1}(a)$ will be called extremal in the problem on the radius of univalence if it is not univalent in any disk $|z| < R$, where $R > r$. The notions of the radii of convexity and starlikeness of the class $H_{(M)}^{*1}(a)$, and of extremal functions in the problems on the radii of convexity and starlikeness, are introduced analogously (cf., for bounded functions, for example, [12]).

Theorem 10. *The extremal function $f^*(z) \in H_{(M)}^{*1}(a)$ in the problem on the radius of univalence of the class $H_{(M)}^{*1}(a)$ has the form:*

$$f^*(z) = \prod\nolimits' \frac{z-\alpha_i}{1-\overline{\alpha_i}\,z} \exp\left[\frac{1}{2\pi}\int_0^{2\pi} \ln\left\{S^{-1}\left[|A|\frac{\prod_1^3|e^{i\theta}-\alpha_i|^2}{|e^{i\theta}-\beta_1|^2\,|e^{i\theta}-\beta_2|^2}\right]\right\}\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\theta\right].$$

Theorem 11. *The extremal function $f^*(z) \in H_{(M)}^{*1}(a)$ in the problem on the radius of starlikeness of the class $H_{(M)}^{*1}(a)$ has the form*

$$f^*(z) = \prod\nolimits' \frac{z-\alpha_i}{1-\overline{\alpha_i}\,z} \exp\left[\frac{1}{2\pi}\int_0^{2\pi} \ln\left\{S^{-1}\left[|A|\frac{\prod_1^3|e^{i\theta}-\alpha_i|^2}{|e^{i\theta}-\beta|^4}\right]\right\}\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\theta\right].$$

Theorem 12. *The extremal function $f^*(z) \in H^{*1}_{(M)}(a)$ in the problem on the radius of convexity of the class $H^{*1}_{(M)}(a)$ has the form*

$$f^*(z) = \prod{}' \frac{z-\alpha_i}{1-\overline{\alpha_i}\, z} \exp\left[\frac{1}{2\pi}\int_0^{2\pi} \ln\left\{S^{-1}\left[|A|\,\frac{\prod_1^4 |e^{i\theta}-\alpha_i|^2}{|e^{i\theta}-\beta|^6}\right]\right\}\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\theta\right].$$

In these theorems α_i and β_i are certain points inside the disk $|z| < 1$. The proof of Theorems 10–12 is based on Theorem 7 and on the methods of [(12)].

We note that a number of the theorems in the paper were formulated using the Luxemburg norm; analogous theorems also hold for the Orlicz norm.

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REFERENCES

[1] I. I. Privalov, *Boundary Properties of Analytic Functions*, 2nd ed., 1950.
[2] Ya. L. Geronimus, Izv. Akad. Nauk SSSR, Ser. Mat., No. 3, 324 (1948).
[3] A. J. Macintyre, W. W. Rogosinski, Acta Math., **82**, 275 (1950).
[4] W. W. Rogosinski, H. Schapiro, Acta Math., **90**, No. 3–4, 287 (1953).
[5] S. Ya. Khavinson, Uchen. zap. Mosk. univ., **148**, 33 (1951).
[6] S. Ya. Khavinson, Mat. sborn., **36**, No. 3, 445 (1955).
[7] G. Ts. Tumarkin, S. Ya. Khavinson, in *Contemporary Problems of the Theory of Functions of a Complex Variable*, 1960, p. 77.
[8] M. A. Krasnosel' skii, Ya. B. Rutitskii, *Convex Functions and Orlicz Spaces*, 1958.
[9] G. P. Safronova, Vestn. LGU, Ser. Mat., Mekh., Astr., No. 13, issue 2, 52 (1959).
[10] N. I. Akhiezer, M. G. Krein, *The L-Problem of Moments*, Moscow; Kharkov, 1938.
[11] S. Ya. Khavinson, Uchen. zap. Vladimirsk. ped. inst. im. Lebedeva-Polyanskogo, issue 2, 43 (1956).
[12] S. Ya. Khavinson, Izvestiya vyssh. uchebn. zaved., Ser. Mat., No. 3(4), 233 (1958).

Note: Figure translations are in progress. See original paper for figures.