Academician of the Academy of Sciences of the Kirghiz SSR M. Ya. LEONOV and L. M. ZORIY

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Fig. 1

Figure 1: Fig. 1

Abstract

Full Text

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ON THE INFLUENCE OF FRICTION ON THE CRITICAL LOAD OF A COMPRESSED ROD

Elastic systems may, as is known ([1]), lose stability of equilibrium as a result of the occurrence of self-oscillations. Investigation of such cases by the method of small oscillations without taking friction into account leads to a certain value of the load parameter P', which divides the interval of variation of the load into a region of instability and a doubtful region. Taking into account small friction, linearly dependent on velocity, one can show that not the whole doubtful region is a region of stability, and that the boundary of the latter sometimes depends essentially on the ratio of the friction coefficients. In connection with the uncertainty of these coefficients, there arises here the problem of determining such a greatest value of the load P_* that, for values of P from the half-interval $[0, P_*)$, the equilibrium of the system is always stable.

In the present paper a solution of the indicated problem is given for the simplest example of a cantilever rod.

Consider an elastic rod of length l with two distributed masses at the free end, compressed by the weight force G of the end masses and by a following force H (Fig. 1). We shall investigate the system with two degrees of freedom (the mass of the rod is neglected). We write the equations of small oscillations of the rod about the rectilinear form of equilibrium. As generalized coordinates we take the deflection v and the angle of rotation φ of the free end. Let us suppose that small friction forces, proportional to the velocity, act in the given system.

Fig. 1

The small oscillations of the system are then described by the equations

$$m\frac{d^2v}{dt^2} = F - b_1\frac{dv}{dt}, \qquad I\frac{d^2\varphi}{dt^2} = M - b_2\frac{d\varphi}{dt}, \tag{1}$$

where F is the force, M the moment, acting on the concentrated masses from the side of the rod; b_1 and b_2 are certain small positive parameters; $I = m\rho^2$ is the central moment of inertia of the end masses, the distance between which is equal to 2ρ.

The quantities F and M are related to the displacement and the angle of rotation of the end of the rod by the relations

$$F = -c_{11}v - c_{12}\varphi, \qquad M = -c_{21}v - c_{22}\varphi, \tag{2}$$

where

$$c_{11} = \frac{G+H}{\Delta} k \sin kl, \qquad c_{12} = \frac{G+H}{\Delta} (\cos kl - 1 + \eta\Delta), \tag{3}$$

$$c_{21} = \frac{G+H}{\Delta} (\cos kl - 1), \qquad c_{22} = \frac{G+H}{\Delta} (\sin kl - kl \cos kl);$$

$$\Delta = 2 - 2\cos kl - kl \sin kl; \tag{4}$$

$$\eta = \frac{H}{G+H}, \qquad k = \sqrt{\frac{G+H}{D}}, \tag{5}$$

D is the flexural rigidity.

Consequently, the equations of small oscillations of the rod can be written as

$$m\frac{d^2v}{dt^2} + b_1\frac{dv}{dt} + c_{11}v + c_{12}\varphi = 0,$$

$$I\frac{d^2\varphi}{dt^2} + b_2\frac{d\varphi}{dt} + c_{21}v + c_{22}\varphi = 0. \tag{6}$$

We seek solutions of the system of equations (6) in the form

$$v = Ae^{\omega t}, \qquad \varphi = Be^{\omega t}, \tag{7}$$

where A, B are constants and ω is a certain parameter. This leads to the characteristic equation

$$p_0\omega^4 + p_1\omega^3 + p_2\omega^2 + p_3\omega + p_4 = 0, \tag{8}$$

where

$$p_0 = Im, \qquad p_1 = b_2 m(\mu\rho^2 + 1), \qquad p_2 = m(\rho^2 c_{11} + c_{22}), \tag{9}$$

$$p_3 = b_2(\mu c_{22} + c_{11}), \qquad p_4 = c_{11}c_{22} - c_{12}c_{21}, \tag{10}$$

$$\mu = b_1/b_2,$$

and the product of the small quantities $b_1 b_2$ is neglected.

The zero solution of the system (6) is asymptotically stable if all roots of equation (8) lie in the left half-plane of the complex variable; according to the Hurwitz theorem ([2]), for this it is necessary and sufficient that all coefficients (9) be positive and satisfy the condition $\Delta_3 > 0$, where

$$\Delta_3 = p_3(p_1p_2 - p_0p_3) - p_1^2 p_4. \tag{11}$$

It is not difficult to show that the quantity (4) $\Delta(kl)$ does not take zero values on the interval $(0, 2\pi)$. Hence it follows that the functions $c_{ij}(kl)$ (3), as well as $p_2(kl)$, $p_3(kl)$, $p_4(kl)$, and $\Delta_3(kl)$, are continuous when $0 < kl < 2\pi$. For the time being we shall restrict ourselves to the indicated interval.

For a sufficiently small compressive force $G + H$, the conditions of the Hurwitz theorem are satisfied and all roots ω have negative real parts. As the load increases, the quantities p_2, p_3, p_4, and Δ_3 vary continuously. In this case instability may occur either when at least one of the roots ω, passing through zero, becomes positive, or when two complex-conjugate roots with negative real parts turn into purely imaginary ones, and then their real parts become positive. On the boundary of the stability region, in the first case, as is known ([2]), the coefficient p_4 vanishes, and in the second case the quantity Δ_3 (11) vanishes. Thus, in the problem under consideration, as the compressive force increases, either p_4 or Δ_3 is the first to change sign, so that it is sufficient to investigate only them.

On the basis of formulas (3) and (4), the coefficient p_4 can be represented as

$$p_4 = \frac{(G+H)^2}{\Delta(\gamma)} \left[\eta + (1-\eta)\cos\gamma\right], \tag{12}$$

where $\gamma = kl$. The smallest positive values of the parameter γ at which p_4 changes sign are denoted by γ_0 and are given in Fig. 2 in the form of a graph.

We proceed to study the quantity Δ_3. Using formulas (9) and (10), one can transform (11) to the form:

$$\Delta_3 = m_0 b_2^2 (\mu\rho^2 + 1)^2 F(R, \rho^2, c_{ij}), \tag{13}$$

Fig. 2

Figure 2: Fig. 2

where

$$F(R, \rho^2, c_{ij}) = R(\rho^2 c_{11} - c_{22})^2 + c_{12}c_{21}; \tag{14}$$

$$R = \mu/(\mu\rho^2 + 1). \tag{15}$$

Obviously, Δ_3 changes sign for the same values of γ as the quantity $F(R, \rho^2, c_{ij})$. Substituting formulas (3) into (14), we obtain

$$F(R, \rho^2, c_{ij}) = \frac{(G+H)^2}{\Delta^2}\{RS^2 + (\cos\gamma - 1)(\cos\gamma - 1 + \eta\Delta)\}, \tag{16}$$

where

$$S = k\rho^2 \sin\gamma - \frac{1}{k}(\sin\gamma - \gamma\cos\gamma). \tag{17}$$

From (16) it is clear that the sign of the function $F(R, \rho^2, c_{ij})$ coincides with the sign of the expression

$$RS^2 + (\cos\gamma - 1)(\cos\gamma - 1 + \eta\Delta). \tag{18}$$

We determine the smallest nonzero values $\gamma_*(\eta)$ for which the quantity (18), or, equivalently, (11), passes from positive to negative values. Since in (18) the first term is nonnegative, and the second is positive for small γ, the values of interest to us will be obtained when the quantity RS^2 is smallest.

Fig. 2

Let us note that the function (15) is always positive and satisfies the condition $R_{\min} < R(\mu) \leq R_{\max}$, where

$$R_{\min} = \lim_{\mu\to 0} R = \lim_{\mu\to\infty} R = 0; \qquad R_{\max} = R\big|_{\mu=\frac{1}{\rho^2}} = \frac{1}{4\rho^2}.$$

Since the ratio μ of the unknown dry-friction coefficients can take any value from the interval $(0, \infty)$, the function $R(\mu)$ can be arbitrarily small. Since the quantity S (17) is bounded for all γ, in (18) the quantity RS^2 can be arbitrarily small. Consequently, $\gamma_*(\eta)$ are the smallest positive roots of the equation

$$(\cos\gamma - 1)[\cos\gamma - 1 + \eta\Delta(\gamma)] = 0. \tag{19}$$

The values $\gamma_*(\eta)$ are shown graphically in Fig. 2.

Let us note that the stability of a cantilever under the action of a follower force depending on the parameter η was considered in (3), where the function determining the dependence of the critical force on η has a point of discontinuity at $\eta = 1/2$. Investigation of the same problem in the formulation proposed here, with friction taken into account, eliminates the indicated discontinuity. The equilibrium of the system is always stable in the region which is shaded in Fig. 2.

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Note: Figure translations are in progress. See original paper for figures.