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1962

Abstract

Full Text

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ON THE INDEX OF SYSTEMS OF TWO-DIMENSIONAL SINGULAR INTEGRAL EQUATIONS

(Presented by Academician I. G. Petrovskii, 20 IX 1961)

In the present paper a formula is derived for the index of systems of two-dimensional singular integral equations, as well as a formula for the index of boundary-value problems for a system of harmonic functions in a three-dimensional domain. The latter is a generalization of the formula obtained in ([1]).

1. We consider a system of singular integral equations

$$a(x)u(x) + \int_S b(x, y-x)u(y)\, d_yS + Tu = f(x), \tag{1}$$

where S is a triply smooth surface, homeomorphic to a sphere, bounding a certain three-dimensional domain D; x, y are points on S; $a(x)$ is a complex square matrix of order p, defined and continuous on S and having continuous derivatives along S up to the second order; $b(x, \alpha)$ is a complex square matrix of order p, defined and continuous for $x \in S$ and arbitrary nonzero vectors $\alpha = (\alpha_1, \alpha_2, \alpha_3)$; T is a regular integral operator. The following conditions are assumed to be fulfilled: 1) $b(x, \alpha)$ has continuous derivatives of arbitrary order with respect to the coordinates of the point α, and is twice continuously differentiable with respect to x along S; 2) $b(x, \rho\alpha) = \rho^{-2}b(x, \alpha)$ $(\rho > 0, \alpha \neq 0)$, as well as the condition for the existence of the singular integral in (1) (see ([2])). The solution u of system (1) is sought in the space $L^2(H)$ of functional columns of height p, whose elements are square-summable along S (satisfy a Hölder condition on S), and the right-hand side f is assumed to belong to the same spaces.

With system (1), in the well-known way (see ([2])), there is associated a symbol $\Phi(\tau)$–a square matrix of order p, defined and continuous on the set P of all unit tangent vectors τ to S. As is known (see ([2])), if the condition

$$\det \Phi(\tau) \neq 0 \quad (\tau \in P) \tag{2}$$

is satisfied, system (1) is normally solvable; the subspaces of solutions of the homogeneous system (1) ($f = 0$) and of the homogeneous adjoint system are finite-dimensional, and the difference $\varkappa = k - k^*$ between the dimensions k and k^* of these subspaces is called the **index** of system (1). S. G. Mikhlin proved that for $p = 1$ the index is equal to zero. In ([1]) it was established that for $p > 1$ the index is, generally speaking, different from zero. In Theorem 1 a formula will be obtained for the index of system (1) for $p > 1$.*

To each matrix $\Phi(\tau)$, defined and continuous on P and satisfying condition (2), there is associated an integer l, defined as follows. Let first $p = 2$, and let $(\Phi_1(\tau) + i\Phi_2(\tau), \Phi_3(\tau) + i\Phi_4(\tau))$ be one of the rows of the matrix $\Phi(\tau)$; $\varphi(\tau) = (\Phi_1(\tau), \Phi_2(\tau), \Phi_3(\tau), \Phi_4(\tau))$. $\psi(\tau) = \varphi(\tau)/|\varphi(\tau)|$, where $|\varphi|$ is the length of the vector φ. ψ maps P into the unit–

* The results presented below were reported by the author at the Fourth All-Union Mathematical Congress in July 1961.

three-dimensional sphere. By definition, $l(\Phi)$ is the degree of this mapping. Obviously, $l(\Phi)$ does not depend on the arbitrary choice of the row of the matrix Φ. For $p > 2$, as is known, the matrix $\Phi(\tau)$ can be continuously deformed, preserving condition (2), into the matrix

$$\begin{pmatrix} E & 0 \\ 0 & \Phi_0(\tau) \end{pmatrix},$$

where E is the identity matrix of order $p-2$, and $\Phi_0(\tau)$ is a matrix of second order. In this case $l(\Phi_0)$ is uniquely determined by the matrix Φ. By definition $l(\Phi) = l(\Phi_0)$.

When condition (2) is satisfied, the following holds.

Theorem 2. *The index ($\varkappa$) of system (1) is computed by the formula*

$$\varkappa = l(\Phi), \tag{3}$$

where Φ is the symbol of this system.

2. Consider the following boundary-value problem: find in the domain D a solution v of the system $\Delta v = 0$, satisfying the boundary condition

$$\lim_{x \to y} B\left(y, \frac{\partial}{\partial x}\right) v(x) = f(y) \qquad (x \in D,\ y \in S), \tag{4}$$

where Δ is the Laplace operator;

$$B\left(y, \frac{\partial}{\partial x}\right) = \sum_{j=1}^{3} B_j(y) \frac{\partial}{\partial x_j};$$

$B_j(y)$ $(j = 1, 2, 3)$ are complex square matrices of order p, defined on S and having second continuous derivatives along S; $f(y) \in H$. The solution v is sought in the class of functional columns of p elements, having second continuous derivatives in D and first continuous derivatives in $D + S$. It is assumed that the condition of Ya. B. Lopatinskii ([3]) is fulfilled:

$$\det B(x, \nu(x) + i\tau) \neq 0 \tag{5}$$

for all $x \in S$ and unit vectors τ tangent to S at the point x. Here $\nu(x)$ is the unit normal vector to S at the point x. It is known that for $p = 1$ the index of problem (4) is equal to zero. For $p > 1$, the following holds.

Theorem 2. *The index* ($\varkappa$) *of problem (4) is computed by the formula*

$$\varkappa = l(B), \tag{6}$$

where B is the matrix occurring in (5).

3. **Proof of the theorems.** Consider the group G of all invertible square matrices of order p, defined and continuous on P, which are symbols of systems of the form (1). To each $\Phi \in G$ associate an integer $\varkappa(\Phi)$, the index of the corresponding system (1). If $\Phi \in G$ and $\Phi_1 \in G$ can be continuously deformed into one another while preserving condition (2), then $\varkappa(\Phi) = \varkappa(\Phi_1)$. Therefore from $l(\Phi) = 0$ it follows that $\varkappa(\Phi) = 0$, and since l and $\varkappa$ homomorphically map G into the additive group of integers, for any $\Phi \in G$

$$\varkappa(\Phi) = \gamma l(\Phi), \tag{7}$$

where γ is a certain constant.

We reduce problem (4) to a system of singular integral equations by means of the simple-layer potential. The symbol of this system is equal to $B(x, \nu(x) + i\tau)$, and therefore, on the basis of (7), the index ($\varkappa$) of problem (4) is $\varkappa = \gamma l(B)$. For a complete proof of the theorems it remains only to show that $\gamma = 1$, and this, evidently, is sufficient to do for $p = 2$. As the boundary operator in (4), take an operator defined by an elliptic first-order system on S with characteristic $\varkappa = 0$ (see ([1])). Then, as shown in ([1]), $\varkappa = 2$. It is verified directly that in this case $l(B) = 2$, whence it follows that $\gamma = 1$.

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Received
15 IX 1961

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Note: Figure translations are in progress. See original paper for figures.