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Abstract

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A BOUNDEDNESS CRITERION FOR SOLUTIONS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS HAVING A HIGHEST TERM

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The present work adjoins (1-6) and, in a certain sense, is the concluding one in this series of papers.

We shall consider the differential equation

$$\frac{\partial^{p_1+p_2+\cdots+p_n} y}{\partial t_1^{p_1} \partial t_2^{p_2} \cdots \partial t_n^{p_n}} - \sum_{(q_1 q_2 \cdots q_n)} A_{q_1 q_2 \cdots q_n} \frac{\partial^{q_1+q_2+\cdots+q_n} y}{\partial t_1^{q_1} \partial t_2^{q_2} \cdots \partial t_n^{q_n}} = x \tag{1}$$

in the domain $0 \leqslant t_1, t_2, \ldots, t_n < \infty$.

Here $y = y(t_1, t_2, \ldots, t_n)$, $x = x(t_1, t_2, \ldots, t_n)$ are vector-functions whose values belong to a Banach (complex) space E; $A_{q_1 q_2 \cdots q_n} = A_{q_1 q_2 \cdots q_n}(t_1, t_2, \ldots, t_n)$ are families of linear operators acting in E. The first term on the left-hand side is assumed to be the "highest" one: this means that $p_j \geqslant q_j$,

$$\sum_1^n p_j > \sum_1^n q_j$$

for every term of

$$\sum_{(q_1 q_2 \cdots q_n)} .$$

The Cauchy-Goursat boundary conditions natural for equation (1) shall, for simplicity and without loss of generality, be considered homogeneous:

$$y\Big|_{t_j=0} = \frac{\partial y}{\partial t_j}\Big|_{t_j=0} = \cdots = \frac{\partial^{p_j-1} y}{\partial t_j^{p_j-1}}\Big|_{t_j=0} = 0 \quad (j = 1, 2, \ldots, n). \tag{2}$$

Suppose further that the coefficients $A_{q_1 \cdots q_n}(t_1, \dots, t_n)$ of equation (1) satisfy the following conditions:

1°. All families $A_{q_1 \cdots q_n}(t_1, \dots, t_n)$ are compact. (For finite-dimensional E this means the uniform boundedness $\|A_{q_1 \cdots q_n}(t_1, \dots, t_n)\|$.)

2°. All $A_{q_1 \cdots q_n}(t_1, \dots, t_n)$ have **weak variation at infinity** (see, for example, (7)). This means that for every $\varepsilon > 0$ there exists $T = T(\varepsilon) > 0$ such that, whenever

$$\sum_1^n t'_j \geqslant T; \qquad \sum_1^n t''_j \geqslant T; \qquad \sum_1^n |t'_j - t''_j| < 1$$

necessarily

$$\|A(t'_1, \dots, t'_n) - A(t''_1, \dots, t''_n)\| < \varepsilon.$$

Let us introduce the following additional notation. Let $A^{(\omega)}$ be some limit operator generated by the family $A(t_1, \dots, t_n)$ as

$$\sum_1^n t_j \to \infty.$$

Under

By $\{A^{(\omega)}_{q_1 \cdots q_n}\}$ we shall mean the totality (corresponding to $\sum_{(q_1 \cdots q_n)}$) of limit operators $A^{(\omega)}_{q_1 \cdots q_n}$, generated by the families $A_{q_1 \cdots q_n}(t_1, \dots, t_n)$ and the common sequences t_j.

Finally, using equation (1) and a certain limiting totality $\{A^{(\omega)}_{q_1 \cdots q_n}\}$, we construct the following operator-function, depending on the complex parameters $\lambda_1, \lambda_2, \dots, \lambda_n$:

$$\Gamma^{(\omega)} = \Gamma^{(\omega)}(\lambda_1, \dots, \lambda_n) = \left(\lambda_1^{p_1} \dots \lambda_n^{p_n} I - \sum_{(q_1 \cdots q_n)} \lambda_1^{q_1} \dots \lambda_n^{q_n} A^{(\omega)}_{q_1 \cdots q_n} \right)^{-1}. \tag{3}$$

A point $(\lambda_1, \dots, \lambda_n)$ at which $\Gamma^{(\omega)}(\lambda_1, \dots, \lambda_n)$ exists and is bounded will be called regular for $\Gamma^{(\omega)}$. Any other point we shall call singular.

In the particular case when (1) is the ordinary differential equation of first order $\dfrac{dy}{dt} - Ay = x$, formula (3) gives $\Gamma^{(\omega)}(\lambda) = (\lambda I - A^{(\omega)})^{-1}$, and the singular points of $\Gamma^{(\omega)}$ coincide with the points of the spectrum of $A^{(\omega)}$.

The main result of the present work is the following

Theorem. *In order that, in the boundary-value problem (1)–(2), to every bounded right-hand side* $\sup \|x(t_1, \dots, t_n)\| < \infty$ *there correspond a bounded solution* $\sup \|y(t_1, \dots, t_n)\| < \infty$*, it is necessary and sufficient that every (i.e., constructed from any limiting totality* $\{A^{(\omega)}_{q_1 \cdots q_n}\}$*) operator-function* $\Gamma^{(\omega)}(\lambda_1, \dots, \lambda_n)$ *have no singular points in the domain* $\operatorname{Re} \lambda_j \geq 0,\ j = 1, 2, \dots, n$*; in other words, that every singular point of* $\Gamma^{(\omega)}$ *have at least one "coordinate" lying in the left (open) half-plane.*

Not being able to present here a detailed proof of the theorem, we give only some of the most essential supporting points.

First of all it is necessary to consider the case when the coefficients $A_{q_1 \cdots q_n}$ are constant (i.e., do not depend on $t_1, \dots, t_n$). The passage from this case to an equation with variable coefficients satisfying conditions 1° and 2° has, however, been set out by us, for an equation of a more special form than (1), in the papers ([3], [6]).

For constant operator coefficients the boundary-value problem (1)–(2) is equivalent to the operator (integral) equation

$$y - \sum_{(q_1 \cdots q_n)} S_1^{p_1 - q_1} \dots S_n^{p_n - q_n} A_{q_1 \cdots q_n} y = S_1^{p_1} \dots S_n^{p_n} x, \tag{4}$$

where

$$S_{jx}(t_1, \dots, t_n) = \int_0^{t_j} x(t_1, \dots, t_{j-1}, s, t_{j+1}, \dots, t_n)\, ds.$$

The solution of equation (4) is given by the following easily verified formula:

$$\begin{aligned} y = \left(\frac{1}{2\pi i}\right)^n \oint_{\gamma_1} \oint_{\gamma_2} \cdots \oint_{\gamma_n} (I - \lambda_1 S_1)^{-1} (I - \lambda_2 S_2)^{-1} \cdots (I - \lambda_n S_n)^{-1} \\ \times \left(I - \sum_{(q_1 \cdots q_n)} \frac{A_{q_1 \cdots q_n}}{\lambda_1^{p_1 - q_1} \dots \lambda_n^{p_n - q_n}} \right)^{-1} S_1^{p_1} \dots S_n^{p_n} x \, \frac{d\lambda_1}{\lambda_1} \cdots \frac{d\lambda_n}{\lambda_n}. \end{aligned} \tag{5}$$

The contours $\gamma_1, \gamma_2, \dots, \gamma_n$ here must be chosen so that every point $(\lambda_1, \lambda_2, \dots, \lambda_n)$, with λ_j lying on γ_j or outside it ($j = 1, 2, \dots, n$), is regular for the operator-function (3).

If one takes into account that

$$(I - \lambda_j S_j)^{-1} S_j^{p_j} = (I - \lambda_j S_j)^{-1} \frac{1}{\lambda_j^{p_j}} - \frac{I + \lambda_j S_j + \cdots + \lambda_j^{p_j - 1} S_j^{p_j - 1}}{\lambda_j^{p_j}},$$

then from (5) it is not difficult to obtain the proof of sufficiency.

A considerably more delicate fact, in our opinion, is necessity. The proof of necessity can also be obtained by means of formula (5); however, in doing so one must use certain nontrivial properties of the operator-function (3), or, equivalently, of the operator-function

$$\Delta(\lambda_1, \dots, \lambda_n) = \left(I - \sum_{(q_1 \dots q_n)} \frac{A_{q_1 \dots q_n}}{\lambda_1^{p_1 - q_1} \dots \lambda_n^{p_n - q_n}} \right)^{-1}.$$

We give one of them—the one most essential for the proof of the theorem and, as it seems to us, of independent interest.

Let $(\lambda_1^0, \dots, \lambda_n^0)$ be a singular point of $\Delta(\lambda_1, \dots, \lambda_n)$ that is a limit point for the set of regular points. Then one of the following two assertions holds.

α) There exists an element $x_0 \in E$, $\|x_0\| = 1$, such that

$$\Delta(\lambda_1, \dots, \lambda_n) x_0 = \frac{\lambda_1 \cdots \lambda_n}{(\lambda_1 - \lambda_1^0) \cdots (\lambda_n - \lambda_n^0)} x_0 -$$

$$-\Delta(\lambda_1, \dots, \lambda_n) \sum_{(k_1 \dots k_m)} \frac{P_{k_1 \dots k_m}(\lambda_1, \dots, \lambda_n)}{(\lambda_{k_1} - \lambda_{k_1}^0) \cdots (\lambda_{k_m} - \lambda_{k_m}^0)} x_0.$$

Here $P_{k_1 \dots k_m}(\lambda_1, \dots, \lambda_n)$ are polynomials in $\lambda_1, \dots, \lambda_n$, and each of the denominators $(\lambda_{k_1} - \lambda_{k_1}^0) \cdots (\lambda_{k_m} - \lambda_{k_m}^0)$ consists of factors $\lambda_1 - \lambda_1^0, \dots, \lambda_n - \lambda_n^0$ taken in an incomplete set (i.e., with $m < n$).

β) There exists a sequence of elements $x_j \in E$, $\|x_j\| = 1$, such that (with the same notation)

$$\Delta(\lambda_1, \dots, \lambda_n) x_j = \frac{\lambda_1 \cdots \lambda_n}{(\lambda_1 - \lambda_1^0) \cdots (\lambda_n - \lambda_n^0)} x_j -$$

$$-\Delta(\lambda_1, \dots, \lambda_n) \sum_{(k_1 \dots k_m)} \frac{P_{k_1 \dots k_m}(\lambda_1, \dots, \lambda_n)}{(\lambda_{k_1} - \lambda_{k_1}^0) \cdots (\lambda_{k_m} - \lambda_{k_m}^0)} x_j + z_j,$$

where $z_j \to \theta$. (In fact, β) contains α) as a special case.)

The structure of

$$\sum_{(k_1 \dots k_m)}$$

is cumbersome in the general case and plays no essential role in the proof. For the case when (1) is a hyperbolic equation of the 2nd order:

$$\frac{\partial^2 y}{\partial t_1\,\partial t_2} - A\frac{\partial y}{\partial t_1} - B\frac{\partial y}{\partial t_2} - Cy = x,$$

relation α) is written as follows:

$$\left(I - \frac{A}{\lambda} - \frac{B}{\mu} - \frac{C}{\lambda\mu}\right)^{-1} x_0 = \frac{\lambda\mu}{(\lambda-\lambda_0)(\mu-\mu_0)} x_0 -$$

$$-\left(I - \frac{A}{\lambda} - \frac{B}{\mu} - \frac{C}{\lambda\mu}\right)^{-1} \left\{\frac{A}{\lambda_0}\frac{\mu}{\mu-\mu_0} + \frac{B}{\mu_0}\frac{\lambda_0}{\lambda-\lambda_0} + \frac{C}{\lambda_0\mu_0}\left(\frac{\mu_0}{\mu-\mu_0} + \frac{\lambda_0}{\lambda-\lambda_0}\right)\right\} x_0.$$

In conclusion we note that the main theorem presented in this paper is readily extended to the case when the boundedness of x and y is understood in the following sense:

$$\sup e^{-\alpha(t_1+\dots+t_n)}\|x(t_1,\dots,t_n)\| < \infty; \qquad \sup e^{-\alpha(t_1+\dots+t_n)}\|y(t_1,\dots,t_n)\| < \infty,$$

where α is a given real number. This makes it possible to obtain from it not only criteria for the boundedness of solutions, but also certain estimates of exponential growth, given in (2^{-6}) for an equation of a more particular form.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.