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Abstract

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E. G. D' yakonov

DIFFERENCE SCHEMES WITH A SPLITTING OPERATOR FOR NONSTATIONARY EQUATIONS

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Let, in the space of x, t $(x = (x_1, x_2, \dots, x_p))$, the solution of a periodic Cauchy problem or of a mixed problem be sought by means of the grid method. Denote by ω_h the grid in x $(h = (h_1, h_2, \dots, h_p))$, i.e., the set of points $(x_s)_{i_s} = i_s h_s$ $(s = 1, 2, \dots, p)$, and by ω_τ the set of points $t_n = \tau_0 + \tau_1 + \dots + \tau_{n-1}$ $(0 \le t_n \le T)$; $Q_{h\tau} = \omega_h \times \omega_\tau$.

Ordinary implicit difference schemes have the form

$$Av^{(n+1)} = F(f^{(n)}, v^{(n)}, v^{(n-1)}, \dots, v^{(n-r)}), \tag{1}$$

where $v^{(n)}$ is the vector of values of the function v on the grid layer $t = t_n$, f is a given function; A, F are certain difference operators depending on $Q_{h\tau}$, with A a p-dimensional operator, i.e., to specify Av_Δ $(\Delta \in \omega_h)$ one must specify the values of v at the points of ω_h belonging to a neighborhood of the point Δ in the space x.

For many problems it is known ([1,2]) that such schemes are absolutely stable, but finding $v^{(n+1)}$ from (1) requires $\sim \dfrac{1}{h^{3p}}$ arithmetic operations.

For one and the same problem there usually exists a class of different implicit absolutely stable difference schemes with the same order of approximation. It therefore seems expedient to single out from this class those difference schemes in which the operator A splits, i.e.,

$$A = \prod_{s=1}^{p} A_s,$$

where A_s are operators of dimension less than p. If all A_s are one-dimensional, then finding $v^{(n+1)}$ from (1) reduces to the successive solution of p systems of equations $A_s w^{(n+s/p)} = w^{(n+(s-1)/p)}$, where $w^{(n)} = F(f^{(n)}, v^{(n)}, \dots, v^{(n-r)})$, $w^{(n+s/p)} = A_{s+1} A_{s+2} \cdots A_p v^{(n+1)}$ $(s = 1, 2, \dots, p-1)$, $w^{(n+p/p)} = v^{(n+1)}$ satisfy the boundary conditions or periodicity conditions for $A_{s+1} A_{s+2} \cdots A_p v^{(n+1)}$ and

$v^{(n+1)}$. In this case the transition from $v^{(n)}$ to $v^{(n+1)}$ is carried out with the expenditure of only $\sim \dfrac{1}{h^p}$ arithmetic operations.

It is interesting to note that both the schemes ([3–6]), proposed for equations with constant coefficients, and their generalizations to the case of equations with separable variables ([7,8]), in those cases when intermediate steps are eliminated, lead to relation (1) with a splitting operator A and an operator F of special form.

In works ([9,10]), for the heat-conduction equation, a difference scheme was proposed in which a factorization of the operator A is carried out. This requirement on A is somewhat more stringent than the requirement of splitting of A, since the factorization of the splitting operator A by the method of ([9,10]) is possible only in the case of commutativity of A_{s_1} and A_{s_2} $(s_1 \neq s_2)$. In the case of an equation with variable coefficients, however, the splitting operator A can be factorized only incompletely ([11]); moreover, because of the incomplete factorization of A, there arises the necessity at each time

layer an iterative process that is a certain modification of the method (12).

In the present note we give some of the difference schemes with a splitting operator that have been obtained for equations with variable coefficients.

1. Let, in the cylinder $Q_T = \overline{\Omega} \times [0 \leq t \leq T]$, the solution of the equation

$$D_0 u = \sum_{s=1}^{p} \left[D_s \left(a_s(x,t) D_{su} \right) + b_s(x,t) D_{su} \right] + d(x,t)u + f, \tag{2}$$

be sought, satisfying the conditions

$$u\Big|_{t=0} = \varphi(x), \qquad u\Big|_{S} = \psi(x,t), \qquad (x,t) \in S, \tag{3}$$

where $D_0 = \partial/\partial t$, $D_s = \partial/\partial x_s$ $(s = 1, 2, \dots, p)$; $\overline{\Omega}$ is a parallelepiped, S is the lateral surface of Q_T; $a_s \geq \gamma^2 > 0$.

Introduce the notation:

$$\Delta_s u = \frac{1}{h_s}\left(u(x_s + h_s) - u(x_s)\right); \qquad \overline{\Delta}_s u = \frac{1}{h_s}\left(u(x_s) - u(x_s - h_s)\right);$$

$$\widetilde{\Delta}_s u = \frac{1}{2}\left(\Delta_s + \overline{\Delta}_s\right) u; \qquad E \text{ is the identity operator;} \qquad L_s^{(n)} = \overline{\Delta}_s a_s^{(n)} \Delta_s + b_s^{(n)} \widetilde{\Delta}_s + d_s^{(n)} E,$$

$$A_s^{(n)} = E - \tau_{nL} s^{(n)}, \qquad a_s^{(n)} = a_s.$$

If Δ is a boundary point of ω_h, then we set

$$v_\Delta^{(n+1)} = \psi_\Delta^{(n+1)}, \tag{4}$$

and if Δ is an interior point of ω_h, then $v_\Delta^{(n+1)}$ is determined from the following system of equations:

$$\frac{1}{\tau_n}\prod_{s=1}^{p} A_s^{(n)} v_\Delta^{(n+1)} = \frac{1}{\tau_n}\left(E+\tau_n \overline{d}^{(n)}\right) v_\Delta^{(n)} + f_\Delta^{(n)}, \tag{5}$$

$$v_\Delta^{(0)} = \varphi_\Delta, \tag{5'}$$

where $d_s, \overline{d}$ are chosen from the conditions: 1) under homogeneous boundary conditions (4) the operator $L_s^{(n)}$ is negative definite; 2)

$$\overline{d}^{(n)} + \sum_{s=1}^{p} d_s^{(n)} = d^{(n)}.$$

Theorem 1. If the coefficients (2) are bounded and $\mathcal{E}^{(n)}$ satisfies (5) with $f=0$ and (4) with $\psi = 0$, then

$$\|\mathcal{E}^{(n+1)}\| \leq (1+c\tau_n)\|\mathcal{E}^{(n)}\|,$$

where $c \geq 0$ and does not depend on τ_n or h for $0 \leq t_n \leq T$;

$$\|v^{(n)}\| = \|v^{(n)}\|_{L_2(\omega_h)}.$$

Theorem 2. Suppose the following conditions are fulfilled:

1) $a_s \in C^3(\overline{\Omega}_t)$, $b_s \in C^0(Q_T)$, $d \in C^0(Q_T)$, $D_s^4 u \in C^0(Q_T)$, $D_0^2 u \in C^0(Q_T)$;
2) for every t $(0 \leq t \leq T)$, $u \in D_2^{(p)}(M(T))$; $D_s a_s, b_s, d_s$, considered as functions of $x_1, x_2, \dots, x_{s-1}, x_{s+1}, \dots, x_p$, belong to $D_2^{(p-1)}(M(T))$.

Then

$$\|u^{(k)} - v^{(k)}\| = O(\|\tau\|) + O(\|h\|^2),$$

where $\overline{\Omega}_{t_0} = Q_T \cap (t=t_0)$, $t_k = \tau_0 + \tau_1 + \dots + \tau_{k-1} \leq T$, $\|\tau\| = \max_n \tau_n$,

$$\|h\| = \max_s h_s,$$

$D_2^{(p)}(M)$ is the set of functions with partial derivatives containing no more than two differentiations with respect to x_s $(s = 1, 2, \dots, p)$, bounded in $\overline{\Omega}$ by the constant M, and u is the solution of (2), (3).

If $a_s = a_s(x_s, t)$, $b_s = 0$, $d = \sum_{s=1}^{p} d_s(x_s, t)$, then for the scheme

$$\frac{1}{\tau_n}\prod_{s=1}^{p}\left(E-\frac{\tau_n}{2}L_s^{(n+1/2)}\right)v^{(n+1)}=\frac{1}{\tau_n}\prod_{s=1}^{p}\left(E+\frac{\tau_n}{2}L_s^{(n+1/2)}\right)v^{(n)}+f^{(n+1/2)}. \quad (6)$$

where $L_s^{(n+1/2)}=\overline{\Delta_s a_s}(x_s,t_n+\tau_n/2)\Delta_s+d_s^{(n)}E$, under somewhat stronger assumptions than in Theorem 2, one can obtain the estimate $\|u^{(k)}-v^{(k)}\|=O(|\tau|^2)+O(|h|^2)$. Note that schemes (5) and (6) can be split by the method of ([5],[8]), if $\psi=0$.

2. For the mixed problem

$$D_0^2u=\sum_{s=1}^{p}\left[D_s(a_s(x_s,t)D_{su})+d_s(x_s,t)u\right]+f(x,t), \quad (7)$$

$$u\,\big|_S=\psi(x,t),\qquad u\,\big|_{t=0}=\varphi(x),\qquad D_0u\,\big|_{t=0}=\varphi_1(x) \quad (8)$$

the difference scheme with splitting

$$A=\prod_{s=1}^{p}(E-\tau^2L_s^{(n)})$$

has the form

$$\frac{1}{\tau^2}Av^{(n+1)}=\frac{2}{\tau^2}\left(A+\frac{1}{2}\sum_{s=1}^{p}L_s^{(n)}\right)v^{(n)}-\frac{1}{\tau^2}Av^{(n-1)}+f^{(n)}; \quad (9)$$

the notation here is the same as in Sec. 1.

Theorem 3. *If the coefficients in (7) are bounded and $\mathcal{E}^{(n)}$ satisfies (9) with $f=0$ and (4) with $\psi=0$, then*

$$\|\mathcal{E}^{(n)}\|\le M(T)\left(\|\mathcal{E}^{(0)}\|+\left\|\frac{\mathcal{E}^{(1)}-\mathcal{E}^{(0)}}{\tau}\right\|\right),\qquad n\tau\le T.$$

Theorem 4. *Let the conditions of Theorem 2 be fulfilled and*

$$D_0^4u\in C^0(Q_T),\qquad D_0^2D_s^{\alpha}u\in C^0(Q_T)\quad(\alpha=0,1,2);$$

$$D_0D_{s_1}^{\alpha_1}D_{s_2}^{\alpha_2}\in C^0(Q_T),\qquad s_1\ne s_2,\quad \alpha_i=0,1,2\ (i=1,2).$$

Then the estimate is valid

$$\|u^{(k)} - v^{(k)}\| = O(\tau^2) + O(h^2).$$

3. Suppose that in the cylinder Q_T (see Sec. 1) one seeks the solution of the periodic Cauchy problem

$$D_0 u = (-1)^{m-1} \sum_{|s|=2m} a_{sD}^{s} u + \sum_{|\alpha|=2m} a_\alpha D^\alpha u + \sum_{|\beta|<2m} a_\beta D^\beta u + f, \qquad (10)$$

$$u \Big|_{t=0} = \varphi(x),$$

where s, α, β are p-dimensional differentiation vectors, and α are the vectors corresponding to mixed derivatives. The coefficients a will be regarded as constant, although some generalizations to the case of variable coefficients are possible; $a_s > 0$.

The difference scheme with splitting A for this problem has the form

$$\frac{1}{\tau_n} \prod_{s=1}^{p} \left(E + \tau_n (-1)^m \widetilde{\Delta}^s + \tau_n d_s\right) v^{(n+1)} =$$

$$= \frac{1}{\tau_n} \left(E + \tau_n \left(\sum_{|\alpha|=2m} a_\alpha \widetilde{\Delta}^\alpha + \sum_{|\beta|<2m} a_\beta \widetilde{\Delta}^\beta + \bar{d}\right)\right) v^{(n)} + f^{(n)}, \qquad (11)$$

$$v_{\Delta+\delta}^{(k)} = v_{\Delta}^{(k)}; \qquad (12)$$

δ is the periodicity vector.

Theorem 5. Suppose: 1)

$$\left|\sum_{|\alpha|=2m} a_\alpha \xi^\alpha\right| \le (1-\sigma) \sum_s a_s |\xi|^s, \quad \xi = (\xi_1, \xi_2, \dots, \xi_p),$$

where $\sigma > 0$, in general, when $\sum_\beta a_\beta^2 > 0$, and $\sigma = 0$ if $\sum_\beta a_\beta^2 = 0$; 2) the d_s are chosen sufficiently large.

Then, if $\mathcal{E}^{(n)}$ satisfies (11) with $f = 0$ and condition (12), then

$$\|\mathcal{E}^{(n+1)}\| \le (1 + c\tau) \|\mathcal{E}^{(n)}\|,$$

where $c \ge 0$ does not depend on τ_n and h ($0 \le n\tau \le T$).

Theorem 6. Suppose: 1) the conditions of Theorem 5 are fulfilled; 2) $u \in C^2(Q_T)$, $u \in D_{2m}^{(p)}(\mu(T))$, $D_s^{2m+2}u \in C^0(Q_T)$.

Then

$$\|u^{(k)} - v^{(k)}\| = O(\|\tau\|) + O(\|h\|^2).$$

Remark 1. Some general criteria for the stability of difference schemes of the form $A_n \mathcal{E}^{(n+1)} = B_n \mathcal{E}^{(n)}$ can be formulated. Namely, for stability with respect to the initial conditions it is sufficient that, under zero boundary conditions or periodicity conditions,

$$\|A_n \mathcal{E}\| \geq (1 - c_1\tau)\|\mathcal{E}\|, \qquad \|B_n\| \leq 1 + c_2\tau.$$

With the aid of this criterion, convergence and the corresponding error estimate for ordinary implicit schemes have been obtained for general domains $\overline{\Omega}$ under weaker smoothness requirements on the solution and on the coefficients of the equation.

Remark 2. Difference schemes with a splitting operator have also been constructed for certain systems of differential equations.

Remark 3. The use of splitting operators in iterative methods makes it possible to obtain useful modifications of methods ([3],[4],[13]).

Moscow State University
named after M. V. Lomonosov

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Note: Figure translations are in progress. See original paper for figures.