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Abstract

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MATHEMATICS

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CYCLOMATIC AND DISTRIBUTIVE PROPERTIES OF MULTIGRAPHS

(Presented by Academician S. L. Sobolev, 11 XII 1961)

1. By a **multigraph** we mean the following generalization of a finite undirected graph, in which loops and parallel edges are allowed (in finite number) ((*3*), Ch. 4). An edge different from a loop is called **normal**. A vertex not incident to any normal edge is called **isolated**. Two multigraphs are **isomorphic** (more precisely, isomorphic up to a permutation of loops) if they possess the same number of loops, and their vertex sets can be put into one-to-one correspondence in such a way that the corresponding pairs of distinct vertices are joined by one and the same number of normal edges; multigraphs are **almost isomorphic** if they become isomorphic after the deletion of all loops and isolated vertices.

Let L be a multigraph with a nonempty and ordered set of normal edges; L_α the multigraph obtained from L by deleting the first normal edge (without deleting vertices); L_β the multigraph obtained from L_α by identifying the ends of the deleted edge (without deleting or identifying other edges).* The order of the remaining normal edges in L_α and L_β is considered the same as in L. A multigraph manifestly having no normal edges will be denoted by G.

If $d_1(L)$ is the number of vertices, $d_2(L)$ the number of edges, $l(L)$ the cyclomatic number, and $x(L)$ the number of connected components of an arbitrary multigraph L, then

$$d_2(L) - l(L) = d_1(L) - x(L) \tag{1}$$

((*3*), Ch. 4), and for $L \neq G$ also

$$\begin{aligned} d_1(L) &= d_1(L_\alpha) = d_1(L_\beta) + 1, \\ d_2(L) &= d_2(L_\alpha) + 1 = d_2(L_\beta) + 1, \end{aligned} \tag{2}$$

$$x(L) = x(L_\beta) = \begin{cases} x(L_\alpha) - 1, & \text{if the first normal edge of } L \text{ is a bridge, i.e. does not enter into any cycle;} \\ x(L_\alpha), & \text{if the first normal edge of } L \text{ is not a bridge.} \end{cases} \tag{3}$$

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Let K be a ring with generators 1 (the identity), α, β; let $\Phi(L) = \Phi(L; \alpha, \beta)$ be a function on multigraphs, with values in K, satisfying the conditions

$$\Phi(L) = \alpha\Phi(L_\alpha) + \beta\Phi(L_\beta) + 1 \quad (L \neq G), \tag{4}$$

$$\Phi(G) = 0.$$

The value $\Phi(L)$, computed on the basis of (4), does not depend on the order of the normal edges of the given multigraph L if and only if the ring K is commutative. Indeed, computing Φ for the two multigraphs shown in Fig. 1 and equating the results, we obtain $\alpha\beta = \beta\alpha$. Conversely, assuming $\alpha\beta = \beta\alpha$, we can prove the coincidence of the values of Φ for any two multigraphs differing only in the order of normal edges, by induction on the number of such edges.

Obviously, the function Φ does not distinguish almost isomorphic multigraphs, and its general expression has the form

$$\Phi(L) = \sum_{i,j>0} \varphi_{ij}(L)\alpha^i\beta^j,$$

where $\varphi_{ij}(L)$ are nonnegative integers, and $\varphi_{ij}(L) = 0$ for $i + j > d_2(L)$.

2. From (4) it follows directly that the function

$$\psi(L) = \psi(L; \alpha, \beta) = 1 + (\alpha + \beta - 1)\Phi(L; \alpha, \beta)$$

satisfies the conditions

$$\begin{aligned} &\psi(L) = \alpha\psi(L_\alpha) + \beta\psi(L_\beta) \quad (L \neq G), \\ &\psi(G) = 1. \end{aligned} \tag{5}$$

Conversely, $\Phi(L)$ is obtained from the general expression $\psi(L; \alpha, \beta)$ by formal division of the polynomial $\psi - 1$ by $\alpha + \beta - 1$. From relations (5) the following properties of the function ψ follow:

A. If the multigraphs L' and L'' are almost isomorphic, then $\psi(L') = \psi(L'')$.

B. Let the multigraph L consist of two multigraphs L' and L'', having not more than one common vertex and not connected with one another by any edge such that both of its ends would be distinct from the common vertex*; then

$$\psi(L) = \psi(L') \cdot \psi(L'').$$

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* In this case, edges parallel to the deleted one turn into loops.

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Fig. 1

Figure 1: Fig. 1

Fig. 1

C. Let L consist of two multigraphs L' and L'', having no common vertices and connected with one another by a single edge (a neck); then

$$\psi(L) = (\alpha + \beta)\psi(L') \cdot \psi(L'').$$

D. If the multigraph L contains no edges other than loops and necks, then

$$\psi(L) = (\alpha + \beta)^{d_2(L) - l(L)}.$$

Denote by K' the commutative ring with generators $1, \alpha, \alpha^{-1}, \beta, u, u^{-1}, v, v^{-1}$ and additional relations

$$\alpha\alpha^{-1} = uu^{-1} = vv^{-1} = 1.$$

Define the function $\psi'(L) = \psi'(L; \alpha, \beta, u, v)$ with values in K' by means of the equalities

$$\begin{aligned} \psi'(L) &= \alpha\psi'(L_\alpha) + \beta\psi'(L_\beta) \quad (L \neq G), \\ \psi'(G) &= u^{d_1(G)} \cdot v^{d_2(G)}. \end{aligned} \tag{6}$$

Obviously, $\psi(L; \alpha, \beta) = \psi'(L; \alpha, \beta, 1, 1)$. On the other hand,

$$\psi'(L; \alpha, \beta, u, v) = u^{d_1(L)} v^{d_2(L)} \psi\left(L; \frac{\alpha}{v}, \frac{\beta}{uv}\right); \tag{7}$$

this follows from (5), taking (2) into account, by replacing α by $\frac{\alpha}{v}$ and β by $\frac{\beta}{uv}$.

From (7), (1), and the properties of the function ψ, it follows, in particular:

A$'$. If the multigraphs L' and L'' are isomorphic, then $\psi'(L') = \psi'(L'')$.

D$'$. If the multigraph L contains no edges other than loops and necks, then

$$\psi'(L; \alpha, \beta, u, v) = (\alpha u + \beta)^{d_2(L) - l(L)} \cdot u^{\chi(L)} v^{l(L)}.$$

3. A **partial multigraph** of L is L itself and any multigraph obtained from L by deleting edges (without deleting vertices). Let $p_{ji}(L)$ be the number of such partial multigraphs of L that have j edges and cyclomatic number i. It is easy to show that

* The common vertex (if it exists) is an articulation point for L ([3], ch. 20).

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$$p_{ji}(L) = p'_{ji}(L_\alpha) + p_{j-1,i}(L_\beta) \quad \text{for } L \neq G, \quad p_{ii}(G) = C^i_{d_2(G)}$$

and $p_{ji}(G) = 0$ for $j \neq i$. Therefore the polynomial

$$P(L) = P(L; x, y) = \sum_{i,k \geq 0} p_{i+k,i}(L) \cdot x^i y^k$$

satisfies the conditions

$$P(L) = P(L_\alpha) + yP(L_\beta) \quad (L \neq G),$$

$$P(G) = (1+x)^{d_2(G)};$$

but these same equations are obtained from (6) by the formal substitution of α by 1, β by y, u by 1, v by $1+x$; consequently,

$$P(L; x, y) = \psi'(L; 1, y, 1, 1+x),$$

or, on the basis of (7),

$$P(L; x, y) = (1+x)^{d_2(L)} \psi\left(L; \frac{1}{1+x}, \frac{y}{1+x}\right). \tag{8}$$

Putting here

$$\frac{1}{1+x} = \alpha, \qquad \frac{y}{1+x} = \beta,$$

we obtain

$$\psi(L; \alpha, \beta) = \alpha^{d_2(L)} \cdot P\left(L; \frac{1}{\alpha} - 1, \frac{\beta}{\alpha}\right). \tag{9}$$

4. Tutte (^{2}) introduced the concept of the **dichromate** of a multigraph L. Without repeating Tutte' s definition, let us list (in our terms) those of the properties of the dichromate proved by him which make it possible to determine it uniquely for any L:

1) The dichromate $\chi(L) = \chi(L; x, y)$ is a polynomial in x, y.
2) If the multigraphs L' and L'' are isomorphic, then $\chi(L') = \chi(L'')$.
3) If the first normal edge of L is not an isthmus, then

$$\chi(L) = \chi(L_\alpha) + \chi(L_\beta).$$

4) If the multigraph L contains no edges other than loops and isthmuses, then

$$\chi(L; x, y) = x^{d_2(L) - l(L)} \cdot y^{l(L)}.$$

Using (3), as well as definition (6) and property Γ' of the function ψ', it is not difficult to verify that the function

$$(x-1)^{-\chi(L)}\psi'(L;1,1,x-1,y)$$

satisfies conditions 1)–4) and, consequently, coincides with the dichromate. Hence, with the help of (7) and (9), we obtain

$$\chi(L;x,y) = (x-1)^{(d_2(L)-l(L)}y^{d_2(L)}\psi\left(L;\frac{1}{y},\frac{1}{(x-1)y}\right) = $$

$$= (x-1)^{d_2(L)-l(L)}P\left(L;y-1,\frac{1}{x-1}\right). \tag{10}$$

5. Let us introduce the **generalized rank polynomial**

$$R(L) = R(L;x,y)$$

of a multigraph L as follows:

$$R(L) = R(L_\alpha) - R(L_\beta) \quad (L \neq G),$$

$$R(G) = y^{d_1(G)} \cdot x^{d_2(G)}. \tag{11}$$

According to this definition,

$$R(L;x,y) = \psi'(L;1,-1,y,x),$$

and hence from (7) and (9) it follows that

$$R(L;x,y) = y^{d_1(L)} \cdot x^{d_2(L)} \cdot \psi\left(L;\frac{1}{x},-\frac{1}{xy}\right) = y^{d_1(L)} \cdot P\left(L;x-1,-\frac{1}{y}\right). \tag{12}$$

Let $r_k(L)$ be the number of ways of coloring the vertices of the multigraph L with k colors subject to the condition that vertices joined by at least one edge must not have the same color; moreover, two colorings are regarded as different when there exist two vertices in L that receive the same color in one coloring and different colors in the other coloring. If L has at least one loop, then $r_k(L) = 0$ for all $k = 1, 2, ...$; hence, also from the expression for $r_k(E_n)$ found in (^{1}) (Chapter 2, §4) for the empty (edgeless) n-vertex graph E_n, it follows that

$$r_k(G) = \frac{\Delta^k O^{d_1(G)}}{k!} \cdot O^{d_2(G)},$$

where $O^{d_2(G)} = 1$ for $d_2(G) = 0$. Further, in exactly the same way as for graphs ((^{1}, Ch. 2, § 3), it is proved that $r_k(L) = r_k(L_\alpha) - r_k(L_\beta)$ for any multigraph $L \neq G$. In view of what has been said, the function $\widetilde{R}(L) = \sum_{k\geq 1} r_k(L) \cdot y^{(k)}$ satisfies the conditions

$$\widetilde{R}(L) = \widetilde{R}(L_\alpha) - \widetilde{R}(L_\beta) \quad (L \neq G),$$

$$\widetilde{R}(G) = y^{d_1(G)} \cdot O^{d_2(G)};$$

therefore, in view of (11), $\widetilde{R}(L) = R(L; 0, y)$. But on the basis of (12) and the definition of $P(L)$ we have

$$R(L; 0, y) = y^{d_1(L)} P\left(L; -1, -\frac{1}{y}\right) = \sum_{i,k \geq 0} (-1)^{i+k} p_{i+k,i}(L) \cdot y^{d_1(L)-k},$$

and finally

$$\sum_{k \geq 1} r_k(L) \cdot y^{(k)} = \sum_{k \geq 0} (-1)^k \left[\sum_{i \geq 0} (-1)^i p_{i+k,i}(L)\right] \cdot y^{d_1(L)-k}, \tag{13}$$

i.e., between the numbers $r_k(L)$ and $p_{ji}(L)$ in the case of multigraphs there hold the same relations as were derived in (4) for graphs.

Tutte (^{2}) investigated the function $\vartheta(L, y)$, connected with the coloring problem, and proved that $\vartheta(L, y) = (-1)^{d_1(L)-\varkappa(L)} \chi(L; 1-y, 0)$. Using (10) and (1), we can express $\vartheta(L, y)$ in terms of $P(L)$, and comparison of this expression with (13) gives

$$\sum_{k \geq 1} r_k(L) \cdot y^{(k)} = y^{\varkappa(L)} \vartheta(L, y),$$

i.e., the function $\vartheta(L, y)$ is completely determined by the number of connected components and by the distribution polynomial ((^{1}, Ch. 2, § 5) of the multigraph L.

6. Let the sets of vertices and edges (including loops) of the multigraph L be numbered, and let $a_{ks}(L) = 1$ if the s-th edge is normal and incident with the k-th vertex, and $a_{ks}(L) = 0$ in all other cases. The incidence matrix modulo two $\|a_{ks}\|$ ($k = 1, 2, \ldots, d_1(L)$; $s = 1, 2, \ldots, d_2(L)$) determines the multigraph L up to a permutation of loops. As is known ((^{3}, Ch. 15), $l(L) = d_2(L) - \operatorname{rank} \|a_{ks}\|$. The same is true for any partial multigraph L'; but the incidence matrix for L' is obtained from the matrix for L by deleting certain columns. Hence it follows that $p_{ji}(L)$ is equal to the number of those matrices, formed from j columns of the matrix $\|a_{ks}\|$, which have rank $j - i$.

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