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Abstract

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MATHEMATICS

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THE CHARACTER OF CERTAIN TYPES OF COMPLEX EQUILIBRIUM STATES IN n-DIMENSIONAL SPACE

(Presented by Academician I. G. Petrovskii on 23 V 1962)

In the present note we establish the possible character of isolated equilibrium states of the system of differential equations:

$$\dot{x}_i = P_i(x_1, x_2, \dots, x_n), \qquad i = 1, 2, \dots, n. \tag{1}$$

($P_i(x_1, x_2, \dots, x_n)$ are analytic functions, or functions possessing a certain finite number of partial derivatives in some domain of the n-dimensional Euclidean space R^n) in the case where one of the roots of the characteristic equation is equal to zero, and the real parts of the remaining $n-1$ roots are different from zero*.

Following Lyapunov, the system (1) in the present case can be brought to the following form:

$$\dot{x}_i = \sum_{j=1}^{n-1} a_{ij}x_j + z^m F_i(z) + \sum_{j=1}^{n-1} x_j f_i(x_1, x_2, \dots, x_{n-1}, z),$$

$$\dot{z} = z^m[\Delta_m + \Phi(z)] + \sum_{j=1}^{n-1} x_j \varphi_j(x_1, x_2, \dots, x_{n-1}, z), \tag{2}$$

$i = 1, 2, \dots, n-1$, where the functions $f_i(x_1, x_2, \dots, x_{n-1}, z)$ and $\varphi_j(x_1, x_2, \dots, x_{n-1}, z)$ vanish at the origin, $F_i(0) = 0$, $\Phi(0) = 0$, $\Delta_m \neq 0$, $m \geq 2$.

The character of the equilibrium state of system (2) has hitherto been considered only in very special cases**. In the present note the possible character of**

equilibrium states of the type under consideration is established in the general case*.

We introduce the following terminology. Let a $(p-1)$-dimensional manifold D, homeomorphic to a $(p-1)$-dimensional sphere, have no contacts with the trajectories of the system. If through each point of the manifold D there passes an O^+-(O^-)-semitrajectory****, then the set of points of these O^+-(O^-)-semitrajectories together with the point O will be called an integral O^+-(O^-)-manifold of dimension p.

Let a $(p-1)$-dimensional manifold γ with boundary, homeomorphic to a $(p-1)$-dimensional hemisphere, have no contacts with the trajectories of the system. If through

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* Such equilibrium states were studied by A. M. Lyapunov ([1]) only from the point of view of stability. The character of the equilibrium state of system (1) has been investigated completely only in the case when the real parts of all roots of the characteristic equation are different from zero ([2]).

** The case where the right-hand side of the last equation depends only on one variable z was considered in ([3]). The case where the right-hand side of the last equation contains no terms of degree lower than m and all nonzero real parts of the roots of the characteristic equation have one and the same sign was considered in ([4]).

*** The present note is a generalization of the results of the author' s work ([5]), obtained for $n = 3$, to the case of an arbitrary number n.

**** A trajectory tending to an equilibrium state is called an O-trajectory (see ([2],[6])). If an O-trajectory tends to the equilibrium state as $t \to +\infty$ $(t \to -\infty)$, then we shall call it an O^+-(O^-)-trajectory. The corresponding semitrajectory will be called an O^+-(O^-)-semitrajectory.

through each point of the manifold γ there passes an O^+ (O^-)-semtrajectory, then the set of these O^+ (O^-)-semtrajectories together with the point O will be called an integral O^+ (O^-)-semimanifold of dimension p.

Lemma 1. *If there exists an O-trajectory of system (2), then it touches either the z-axis or the hyperplane $z = 0$.*

Let k be the number of roots of the characteristic equation with negative real part and, correspondingly, $n-1-k$ the number of roots with positive real part.

The following lemma establishes the dimension of the set of all O-trajectories that do not touch the z-axis.

Lemma 2. *All O^+-trajectories that do not touch the z-axis fill an integral manifold of dimension k, and all O^--trajectories that do not touch the z-axis fill a manifold of dimension $n-1-k$.*

Let $x_1, \ldots, x_k$ be the coordinates corresponding to the roots of the characteristic equation with negative real parts, and, correspondingly, $x_{k+1}, \ldots, x_{n-1}$ the coordinates corresponding to roots with positive real parts. Put

$$\rho_1^2 = \sum_1^k x_i^2; \qquad \rho_2^2 = \sum_{k+1}^{n-1} x_i^2.$$

Lemma 3. *The hypersurfaces $\rho_1^2 + z^{2m} = \rho_2^2$ and $\rho_2^2 + z^{2m} = \rho_1^2$, in a sufficiently small neighborhood of the equilibrium state, are hypersurfaces without contact for the trajectories of system (2).*

Lemma 4. *If there exists an O-trajectory touching the z-axis, then along it the expressions $\rho_1(t)/z^{m-1}(t)$ and $\rho_2(t)/z^{m-1}(t)$ tend to zero.*

We shall distinguish the following three cases: 1) m is odd, $\Delta_m < 0$; 2) m is odd, $\Delta_m > 0$; 3) m is even.

Lemma 5. *In the case when m is odd and $\Delta_m < 0$, the hypersurfaces $\rho_1^2 + z^{m+1} = c_i$ (c_i sufficiently small) in the region bounded by the inequality $\rho_2^2 \leq \rho_1^2 + z^{2m}$ have no contacts with the trajectories of system (2).*

Lemma 6. *In the case when m is odd and $\Delta_m > 0$, the hypersurfaces $\rho_2^2 + z^{m+1} = c_i$ (c_i sufficiently small) in the region bounded by the inequality $\rho_1^2 \leq \rho_2^2 + z^{2m}$ have no contacts with the trajectories of system (2).*

The following lemma, in the case of odd m, establishes the existence of O-trajectories touching the z-axis, and also establishes the dimension of the set of these trajectories.

Lemma 7. *In the case when m is odd and $\Delta_m < 0$ ($\Delta_m > 0$), there exist O^+ (O^-)-trajectories of system (2) touching the z-axis, and all of them fill "almost all" of an integral O^+ (O^-)-manifold of dimension $k+1$ ($n-k$).*

Lemma 8. *In the case when m is even, there exists an integral manifold S of dimension $n-1$, to which the equilibrium state O and all O-trajectories not touching the z-axis belong, such that, in a sufficiently small neighborhood of the equilibrium state, the semiaxis $z > 0$ lies on one side of it, and the semiaxis $z < 0$ on the other.*

One can specify a sufficiently small closed neighborhood δ, containing the equilibrium state, such that the $(n-1)$-dimensional manifold S divides this neighborhood into two parts.

Denote by δ^+ (respectively δ^-) the closure of that part of the neighborhood δ which contains the part of the semiaxis $z > 0$ (respectively $z < 0$).

Lemma 9. *In the case when m is even and $\Delta_m < 0$ ($\Delta_m > 0$): 1) the hypersurfaces $\rho_1^2 + z^m = c_i$ in the part of the closed region δ^+ (δ^-), bounded by the inequality $\rho_2^2 \leq \rho_1^2 + z^{2m}$, have no contacts with the trajectories of system (2); 2) the hypersurfaces $\rho_2^2 + z^m = c_i$ in the part of the closed region δ^- (δ^+),* *bounded by the inequality $\rho_1^2 \leq \rho_2^2 + z^{2m}$, have no contacts with the trajectories of system (2).*

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* The term "almost all" manifold of dimension p means: the whole manifold, except for a manifold of dimension less than p (see (6)).

The following lemma, in the case of even m, establishes the existence of O-trajectories tangent to the z-axis, and also establishes the dimension of the set of these trajectories.

Lemma 10. *In the case when m is even, there exist O^+- and O^--trajectories of system (2) tangent to the z-axis; moreover, all O^+-trajectories fill almost all of the integral O^+-semimanifold of dimension $k+1$, and all O^--trajectories fill almost all of the integral O^--semimanifold of dimension $n-k$.*

The following theorem characterizes the behavior of an individual trajectory passing through a sufficiently small neighborhood of the equilibrium state*:

Theorem 1. *If one of the roots of the characteristic equation is equal to zero, and the real parts of the remaining roots are nonzero, then every trajectory passing through a sufficiently small neighborhood of the equilibrium state is either an O-trajectory or leaves this neighborhood (i.e., is a saddle trajectory).*

The behavior of the whole set of trajectories in a sufficiently small neighborhood of the equilibrium state is described by the following theorem:

Theorem 2. *If one of the roots of the characteristic equation of system (2) is equal to zero, k roots have negative real parts, and $n-1-k$ roots have positive real parts, then the following two cases may occur:*

1) *m is odd, $\Delta_m < 0$ ($\Delta_m > 0$). There exists a unique integral O^+-manifold of dimension $k+1$ (k) and a unique integral O^--manifold of dimension $n-1-k$ ($n-k$).*
2) *m is even. There exists a unique integral O^+-semimanifold of dimension $k+1$ and there exists a unique integral O^--semimanifold of dimension $n-k$. All trajectories not belonging to these two semimanifolds are saddle trajectories.*

Thus, in the case of odd m, the equilibrium state is a **generalized node** if $k=0$, $\Delta_m > 0$ or $k=n-1$, $\Delta_m < 0$, and the equilibrium state is a **generalized saddle** if these conditions are not satisfied.

In the case of even m, by analogy with the corresponding equilibrium state in three-dimensional space (see (5)), it is natural to call it a **generalized saddle-node** if $k=0$ or $k=n-1$, and a **generalized equilibrium state of type** C if $k \neq 0$ and $k \neq n-1$.

In the absence of multiple roots of the characteristic equation, one can formulate the following theorem, which establishes the manner in which O-trajectories approach the equilibrium state:

Theorem 3. *If one of the roots of the characteristic equation is equal to zero, and all the remaining roots are real and distinct, every O-trajectory is a regular O-trajectory**. If the characteristic equation has at least one pair of complex roots, then the system has both regular and singular O-trajectories.*

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* This theorem is analogous to the theorem of I. G. Petrovskii proved for the case when the real parts of all roots of the characteristic equation are nonzero (see (2,6)).

** An O-trajectory having a definite tangent at the equilibrium state O is called a regular O-trajectory; otherwise it is a singular O-trajectory (see (6)).

Note: Figure translations are in progress. See original paper for figures.