ON THE BEHAVIOR OF THE GREEN FUNCTION OF A MULTIPOINT LINEAR DIFFERENCE BOUNDARY-VALUE PROBLEM

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Abstract

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In recent years a number of reports at the Izhevsk Mathematical Seminar were devoted to the question of the behavior of the Green function of certain boundary-value problems for differential and difference equations. Knowing the behavior of the Green function for the corresponding linear boundary-value problem, one can establish theorems of the type of S. A. Chaplygin' s theorem on differential inequalities ([1]). Linear theorems, in turn, make it possible to obtain, for certain nonlinear boundary-value problems, existence theorems and estimates of solutions, if one uses the general ideas of the works ([2], [3]).

In the works ([4], [5]) the question of the sign of the Green function of a two-point boundary-value problem for a second-order difference equation was considered.

In the present note, Maman' s theory ([6]) is carried over to difference equations, and on the basis of the results obtained the behavior of the Green function of the n-point difference boundary-value problem is investigated

$$\mathcal{L}'[u] \equiv u(x+n) + \sum_{k=0}^{n-1} p_k(x) u(x+k) = \varphi(x),$$

$$u(x_i) = a_i \qquad (i = 1, \dots, n) \tag{1}$$

(x_i are integers, $0 = x_1 < x_2 < \dots < x_n = r$; a_i are given numbers), where the functions $\varphi(x)$ and $p_k(x)$ $(k = 0, 1, \dots, n-1)$ are defined for $x = 0, 1, \dots, r-n$.

Here, by the Green function of problem (1) we mean the function $G(x, s)$, defined for $x = 0, 1, \dots, r$; $s = 0, 1, \dots, r-n$, and, for any fixed s, satisfying the boundary-value problem:

$$\mathcal{L}[G] = \delta_{x,s} \quad (\delta_{x,s} = 0 \text{ for } x \neq s,\ \delta_{s,s} = 1),$$

$$G(x_i, s) = 0 \qquad (i = 1, \dots, n).$$

Let the function $f(x)$ be defined on the set $g: \ x = 0, 1, \dots, r \geq n$. We shall agree to assign to the value $f(x_0) = 0$ $(x_0 \in g)$ the sign opposite to the sign of $f(x_0 - 1)$, if only $x_0 \geq 1$; otherwise we assign to the value $f(x_0) = 0$ the sign opposite to the sign of $f(x_0 + 1)$. We shall say that $f(x)$ has a change of sign at the point $x^* \in g$, if $x^* \leq r - 1$ and the signs of $f(x^*)$ and $f(x^* + 1)$, determined with account of the above rule, are opposite.

We shall call the equation

$$\mathcal{L}[u] \equiv u(x+n) + \sum_{k=0}^{n-1} p_k(x) u(x+k) = 0 \tag{2}$$

nonoscillatory on the set g, if every nontrivial solution of it has on this set no more than $n - 1$ changes of sign.

Denote by T the operator defined by the equality

$$Tu(x) = u(x+1).$$

Theorem 1. The difference operation $\mathcal{L}[u]$ can be represented in the form of a product of first-order operations

$$\mathcal{L}[u] \equiv [T - a_n(x)] \cdots [T - a_1(x)] u(x) \tag{3}$$

with coefficients $a_i(x) > 0$ $(x = 0, 1, \dots, r - i; \ i = 1, \dots, n)$ if and only if equation (2) is nonoscillatory on the set g.

The proof of this theorem is divided into a number of lemmas, given below.

For a system of functions $u_k(x)$ $(k = 1, \dots, n)$, defined on the set g, introduce the notation: $D[u_1] = u_1(x)$,

$$D[u_1, \dots, u_k] = \begin{vmatrix} u_1(x) & \cdots & u_k(x) \\ \cdots & \cdots & \cdots \\ u_1(x+k-1) & \cdots & u_k(x+k-1) \end{vmatrix} \qquad (k = 2, \dots, n).$$

Lemma 1. In order that the operation $\mathcal{L}[u]$ be representable in the form (3), it is necessary and sufficient that there exist a fundamental system of solutions $u_k(x)$ $(k = 1, \dots, n)$ of equation (2) for which each of the determinants $D[u_1, \dots, u_k]$ preserves its sign for $x = 0, 1, \dots, r - k + 1$ $(k = 1, \dots, n)$.

Lemma 2. Let $u_k(x)$ $(k = 1, \dots, n)$ be a system of functions defined on the set g and satisfying the conditions

$$u_k(j) = 0 \ (j = 0, 1, \dots, n - k - 1), \qquad u_k(n - k) = 1 \ (k = 1, \dots, n); \tag{4}$$

let $v_k(x)$ $(k = 1, \dots, n)$ be the functions defined by the equalities

$$v_k(x) = \sum_{i=k}^{n} C_{n-k}^{n-i} \alpha^{n-i} u_i(x) \quad (k = 1, \dots, n-1), \qquad v_n(x) = u_n(x),$$

where $\alpha > 0$ is some number.

If each of the determinants $D[u_1, \dots, u_k]$ preserves its sign for $x = n-k, n-k+1, \dots, r-k+1$ $(k = 1, \dots, n)$, then, for sufficiently large α, each of the determinants $D[v_1, \dots, v_k]$ preserves its sign for $x = 0, 1, \dots, r-k+1$ $(k = 1, \dots, n)$.

Lemma 3. Let $u_k(x)$ $(k = 1, \dots, n)$ be a fundamental system of solutions of equation (2) satisfying the initial conditions (4).

If each of the determinants $D[u_1, \dots, u_k]$ preserves its sign for $x = n-k, n-k+1, \dots, r-k+1$ $(k = 1, \dots, n)$, then the operation $\mathcal{L}[u]$ can be represented in the form (3).

Lemma 4. Let $u_k(x)$ $(k = 1, \dots, n)$ be a fundamental system of solutions of equation (2) satisfying the initial conditions (4).

If equation (2) is nonoscillatory on the set g, then each of the determinants $D[u_1, \dots, u_k]$ preserves its sign for $x = n-k, n-k+1, \dots, r-k+1$ $(k = 1, \dots, n)$.

Lemma 5. If a function $v(x)$, defined on the set g, has changes of sign at points x_1 and x_2 of this set $(x_1 < x_2)$, then the function

$$\psi(x) = v(x+1) - a(x)v(x), \tag{5}$$

where $a(x) > 0$ $(x = x_1, x_1+1, \dots, x_2)$, has a change of sign at least at one of the points $x = x_1, x_1+1, \dots, x_2-1$.

Corollary 1. If a function $v(x)$ has m changes of sign on the set g, then the function $\psi(x)$, defined by equality (5), where $a(x) > 0$ $(x = 0, 1, \dots, r-1)$, has at least $m-1$ changes of sign on the set $x = 0, 1, \dots, r-1$.

Corollary 2. Let $\varphi_i(x)$ $(i = 0, 1, \dots, n)$ be functions defined by the equalities

$$\varphi_0(x) \equiv v(x), \qquad \varphi_i(x) = \varphi_{i-1}(x+1) - a_i(x)\varphi_{i-1}(x)$$

$$(a_i(x) > 0;\ x = 0, 1, \dots, r-i;\ i = 1, \dots, n).$$

If $v(x)$ has m sign changes on the set g, then $\varphi_i(x)$ has at least $m-i$ sign changes on the set $x = 0, 1, \dots, r-i$ $(i = 1, \dots, n)$.

Lemma 6. If the function $\varphi(x)$ preserves its sign for $x = 0, 1, \dots, r-k$, then any solution of the equation

$$[T - a_k(x)] \dots [T - a_1(x)] v(x) = \varphi(x)$$

$$(a_i(x) > 0;\ x = 0, 1, \dots, r - i;\ i = 1, \dots, k)$$

has no more than k sign changes on the set g.

Lemma 7. If the operation $\mathcal{L}[u]$ can be represented in the form (3), then equation (2) is nonoscillatory on the set g.

Denote by g_i the set of points $x = x_i + 1, x_i + 2, \dots, x_{i+1} - 1$ $(i = 1, 2, \dots, n-1)$, where $x_i \in g$ $(i = 1, \dots, n)$, $0 < x_1 < x_2 < \dots < x_n = r$.

Theorem 1 and Lemmas 5, 6 make it possible to prove the following assertion.

Theorem 2 (cf. [7]). If equation (2) is nonoscillatory on the set g, then the Green' s function $G(x, s)$ of problem (1) exists and satisfies the relations

$$\operatorname*{sgn}_{\substack{x \in g_i \\ s = 0, 1, \dots, r - n}} G(x, s) = (-1)^{n-i} \qquad (i = 1, 2, \dots, n - 1)$$

everywhere where $G(x, s) \neq 0$.

Corollary 1. Let n be even and let equation (2) be nonoscillatory on the set g.

If $x_i = x_{i+1} - 1$ for $i = 2, 4, \dots, n-2$, then $G(x, s) \leqslant 0$ $(x \in g,\ s = 0, 1, \dots, r-n)$.

If $x_i = x_{i+1} - 1$ for $i = 1, 3, \dots, n-1$, then $G(x, s) \geqslant 0$ $(x \in g,\ s = 0, 1, \dots, r-n)$.

Corollary 2. Let n be odd and let equation (2) be nonoscillatory on the set g.

If $x_i = x_{i+1} - 1$ for $i = 1, 3, \dots, n-2$, then $G(x, s) \leqslant 0$ $(x \in g,\ s = 0, 1, \dots, r-n)$.

If $x_i = x_{i+1} - 1$ for $i = 2, 4, \dots, n-1$, then $G(x, s) \geqslant 0$ $(x \in g,\ s = 0, 1, \dots, r-n)$.

Theorem 2 naturally leads to the question of conditions under which equation (2) is nonoscillatory on the set g. From Lemmas 3, 7, and 4 there follows, obviously, the following criterion for nonoscillation of equation (2).

Theorem 3. Let $u_k(x)$ $(k = 1, \dots, n)$ be a fundamental system of solutions of equation (2), satisfying the initial conditions (4).

For equation (2) to be nonoscillatory on the set g, it is necessary and sufficient that each of the determinants $D[u_1, \dots, u_k]$ preserve its sign respectively on the set $x = n - k, n - k + 1, \dots, r - k + 1$ $(k = 1, \dots, n)$.

To apply this criterion it is necessary to know the values of each of the solutions $u_k(x)$ $(k = 1, \dots, n)$ at all points of the set g. For $r < \infty$ the mentioned values can always be computed from equation (2).

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Note: Figure translations are in progress. See original paper for figures.