M. V. MASLENNIKOV

1962

Abstract

Full Text

MATHEMATICAL PHYSICS

M. V. MASLENNIKOV

ON AN INVERSE PROBLEM IN THE THEORY OF THE PASSAGE OF RADIATION THROUGH MATTER

(Presented by Academician M. V. Keldysh, March 29, 1962)

One of the basic characteristics of the interaction of monochromatic radiation with matter is the scattering indicatrix $g(x)$. The quantity $2\pi g(x)\,dx$ is the probability that, as a result of a single scattering, a light quantum will deviate by an angle χ from the initial direction of motion, where $x < \cos\chi < x + dx$. The number

$$c = 1 - 2\pi \int_{-1}^{1} g(x)\,dx$$

is the corresponding probability of absorption.

In the theory of the passage of radiation through thick plane layers of matter it is known that, far from the boundary of the layer, the principal part of the spatial-angular distribution of the radiation density is uniquely determined by specifying a parameter $\lambda \in (0,1)$ and a nonnegative function $\varphi(\mu)$, connected with the indicatrix $g(x)$ by the equation

$$(1 + \lambda\mu)\,\varphi(\mu) = \int_{-1}^{1} g(\mu, \mu')\,\varphi(\mu')\,d\mu', \tag{1}$$

where

$$g(\mu, \mu') = \int_{0}^{2\pi} g\Big(\mu\mu' + \sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos\varphi\Big)\,d\varphi. \tag{2}$$

For example, in the problem of radiation diffusion in a medium filling the half-space $\tau > 0$, the asymptotic form of the radiation density for large values of the optical thickness τ has the form $\exp(\lambda\tau)\varphi(\mu)$. Here λ and $\varphi(\mu)$ are determined by equation (1), and μ is the cosine of the angle between the direction of propagation of the radiation and the inward normal to the boundary of the half-space.

The following inverse problem in transport theory is of interest: from the asymptotic characteristics of the radiation density (i.e., from a given function $\varphi(\mu)$ satisfying some equation of the form (1)), to reconstruct the indicatrix $g(x)$. It is not difficult to show that, for unique solvability of this problem, at least a small absorption of radiation by the medium is necessary ($c > 0$).

Let Γ be the class of all indicatrices $g(x)$, each of which has the following properties:

1) $g(x)$ is defined and nonnegative on $[-1, 1]$; $g \in L_2(-1, 1)$, and

$$2\pi \int_{-1}^{1} g(x)\, dx = g_0, \qquad 0 < g_0 < 1.$$

As shown in ([1]), for each $g \in \Gamma$ equation (1) admits only one (up to normalization of φ) nontrivial solution $(\lambda_0(g), \varphi^{[g]}(\mu))$ in the class of pairs (λ, φ) with $\lambda \in (0, 1)$, $\varphi \geq 0$, $\varphi \in L_2$*. In what follows the functions $\varphi^{[g]}$ will be assumed normalized by the condition

$$\int_{-1}^{1} \varphi^{[g]}(\mu)\, d\mu = 2.$$

* Condition 2) in paper ([1]) is superfluous.

Now our goal is to establish that different functions g of the class Γ correspond to different solutions $\varphi^{[g]}(\mu)$ of equation (1). To shorten the subsequent arguments we introduce the following notation. Denote the mapping constructed above, $g \to \varphi^{[g]}$, by $\tilde{\omega}_0$: $\omega_0 g = \varphi^{[g]}$, and let Ω_0 be the class of all functions $\varphi^{[g]}$, $g \in \Gamma$. According to this definition, $\tilde{\omega}_0$ turns out to be a one-to-one mapping of Γ onto Ω_0, and Ω_0 is the class of all nonnegative solutions φ of equation (1) for $g \in \Gamma$, normalized by the condition

$$\int_{-1}^{1} \varphi(\mu)\, d\mu = 2.$$

Theorem 1. $\tilde{\omega}_0$ maps Γ onto Ω_0 one-to-one.

This is the theorem of existence and uniqueness of the solution of the inverse problem. By the very formulation of the problem, only the assertion of uniqueness of the solution is nontrivial. Note that, according to theorem 1, the indicatrix $g \in \Gamma$ and the parameter $\lambda = \lambda_0(g)$ are uniquely determined by the specification of just one function $\varphi \in \Omega_0$.

The proof of theorem 1 is achieved by a partial description of the class Ω_0. Let $g \in \Gamma$. Then $g(x)$ admits an expansion in a series, convergent in the mean, in Legendre polynomials

$$g(x) = \sum_{s=0}^{\infty} \frac{2s+1}{4\pi} g_s P_s(x), \tag{3}$$

where

$$\sum_{s=0}^{\infty} s g_s^2 < \infty. \tag{4}$$

Theorem 2. Let $\varphi \in \Omega_0$. Then φ is a positive continuous function of $\mu \in [-1, 1]$,

$$\int_{-1}^{1} \varphi(\mu)\, d\mu = 2.$$

Let φ_k be the coefficients of the expansion of φ in a series in Legendre polynomials,

$$\varphi_k = \frac{2k+1}{2} \int_{-1}^{1} \varphi(\mu) P_k(\mu)\, d\mu,$$

so that

$$\varphi(\mu) = \sum_{k=0}^{\infty} \varphi_k P_k(\mu),$$

and the series converges at least in the mean.

The coefficients φ_k form an alternating-sign sequence, all elements of which are nonzero:

$$\varphi_k \neq 0; \qquad \operatorname{sgn} \varphi_k = (-1)^k, \qquad k = 0, 1, 2, \ldots; \qquad \varphi_0 = 1.$$

There exists the limit

$$\lim_{k=\infty} \frac{\varphi_{k-1}}{\varphi_k} = -\alpha.$$

Moreover, $\alpha > 1$ and

$$\frac{\varphi_{k-1}}{\varphi_k} = -\alpha \left(1 - \frac{1}{2k}\right) + \delta_k,$$

where

$$\sum_{k=1}^{\infty} k\delta_k^2 < \infty.$$

Let $g \in \Gamma$ and $\breve{\omega}_0 g = \varphi$. Then

$$\lambda_0(g) = \frac{2\alpha}{\alpha^2+1},$$

$$g_k = 1 + \lambda_0(g)\left[\frac{k}{2k-1}\frac{\varphi_{k-1}}{\varphi_k} + \frac{k+1}{2k+3}\frac{\varphi_{k+1}}{\varphi_k}\right], \quad k = 0, 1, 2, \dots; \quad \varphi_{-1} = 0. \quad (5)$$

It is obvious that Theorem 1 is an immediate consequence of Theorem 2. For the proof of Theorem 2, equation (1) is reduced to an infinite system of algebraic equations connecting the coefficients φ_k and g_k. The latter is investigated by the method of continued fractions.

Theorem 2 does not give a complete description of the class Ω_0. However, if some function $\varphi(\mu)$ possesses all the properties listed in Theorem 2, then the coefficients g_k, computed by formulas (5), will satisfy condition (4). This makes it possible to determine uniquely the function $g(x) \in L_2$ in accordance with relation (3). The functions $\varphi(\mu)$ and $g(x)$ will be related to one another by equation (1) under condition (2), $\lambda = \frac{2\alpha}{\alpha^2+1}$, $\alpha = \lim_{k=\infty} \frac{\varphi_{k-1}}{\varphi_k}$. Thus $g(x)$ plays the role of an indicatrix, although, generally speaking, it does not belong to the class Γ.

I express my gratitude to A. N. Tikhonov for posing the question and for useful discussion.

Received
27 III 1962

CITED LITERATURE

1. M. V. Maslennikov, DAN, **118**, No. 5, 895 (1958).

Note: Figure translations are in progress. See original paper for figures.