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Abstract

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B. L. ROZHDESTVENSKII

CONSTRUCTION OF DISCONTINUOUS SOLUTIONS OF A SYSTEM OF TWO QUASILINEAR EQUATIONS

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For a system of two quasilinear equations of hyperbolic type

$$\frac{\partial u_i}{\partial t} + \frac{\partial \varphi_i(u)}{\partial x} = 0; \qquad u = \{u_1, u_2\} \quad (i = 1, 2) \tag{1}$$

a method is indicated for constructing generalized discontinuous solutions. Under certain restrictions on the class of systems (1), this method leads to the construction of a generalized discontinuous solution; the uniqueness of the latter is established.

1. Restrictions on the class of systems (1). Let the functions $\varphi_i(u)$ possess continuous second derivatives with respect to their arguments. As is known (^{1,2}), system (1) can be reduced to the form

$$\frac{\partial r_k}{\partial t} + \xi_k(r) \frac{\partial r_k}{\partial x} = 0 \quad (k = 1, 2), \tag{2}$$

where $r_k = r_k(u)$ are the Riemann invariants. We shall assume that $\xi_1(r) < \xi_2(r)$. We shall require of system (1) that

$$\frac{\partial \xi_k(r)}{\partial r_k} > 0 \quad (k = 1, 2). \tag{3}$$

Let $u^0 = \{u_1^0, u_2^0\}$ and $u = \{u_1, u_2\}$ be the values of the solution on different sides of the discontinuity line $x = Dt$. Then, as is known (^{1,2}), these quantities are related by the Hugoniot conditions

$$D(u_i - u_i^0) = \varphi_i(u) - \varphi_i(u^0) \quad (i = 1, 2). \tag{4}$$

Fig. 1

Figure 1: Fig. 1

Fig. 1

Eliminating the quantity D from the two equations (4), we obtain one equation relating the quantities u, u^0. We shall regard the point u^0 as fixed. Then, as is known ($^{1-3}$), the equation obtained determines two smooth curves passing through the point u^0 and tangent at this point to the lines $r_2 = r_2^0$, $r_1 = r_1^0$ (Fig. 1).

We shall assume that the line tangent to the straight line $r_2 = r_2^0$ is projected one-to-one onto the straight lines $r_2 = \text{const}$, and the line tangent to the straight line $r_1 = r_1^0$ is projected one-to-one onto the straight lines $r_1 = \text{const}$. Thus, our second assumption means that equations (4) can be written in the form

$$r_2 = R_2(r_1, r_1^0, r_2^0); \qquad D = D_1(r_1, r_1^0, r_2^0); \tag{5}$$

$$r_1 = R_1(r_2, r_1^0, r_2^0); \qquad D = D_2(r_2, r_1^0, r_2^0); \tag{6}$$

moreover, the functions R_2, D_1, R_1, D_2 are single-valued and possess continuous first derivatives with respect to their variables.

The following properties of these functions are known ($^{2-4}$):

$$R_2(r_1^0, r_1^0, r_2^0) = r_2^0; \qquad D_1(r_1^0, r_1^0, r_2^0) = \xi_1(r_1^0, r_2^0), \tag{7}$$

$$R_1(r_2^0, r_1^0, r_2^0) = r_1^0; \qquad D_2(r_2^0, r_1^0, r_2^0) = \xi_2(r_1^0, r_2^0).$$

We shall require that:

$$\text{for } r_1 < r_1^0 \quad \xi_1(r_1, R_2(r_1, r_1^0, r_2^0)) < D_1(r_1, r_1^0, r_2^0) < \xi_1(r_1^0, r_2^0), \tag{8}$$

for $r_2 > r_2^0$

$$\xi_2(R_1(r_2, r_1^0, r_2^0), r_2) > D_2(r_2, r_1^0, r_2^0) > \xi_2(r_1^0, r_2^0), \tag{9}$$

$$\left|\partial R_2(r_1, r_1^0, r_2^0)/\partial r_1\right| < 1; \qquad \left|\partial R_1(r_2, r_1^0, r_2^0)/\partial r_2\right| < 1.$$

The listed conditions are satisfied by systems of type (A), considered in papers (^{1,4}), by systems of type V in paper (^{3}), and, in particular, by the systems of equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v}{\partial x} = 0; \qquad \frac{\partial \rho v}{\partial t} + \frac{\partial}{\partial x}\left[p(\rho) + \rho v^2\right] = 0;$$

$$\frac{\partial \eta}{\partial t} - \frac{\partial v}{\partial x} = 0; \qquad \frac{\partial v}{\partial t} + \frac{\partial}{\partial x} p\left(\frac{1}{\eta}\right) = 0, \tag{10}$$

which are important for applications, in the case $p' > 0,\ p'' \geq 0$.

2. The method of characteristics. The construction of a classical solution of the Cauchy problem for systems of quasilinear equations by the method of characteristics is presented in papers (^{5,6}). In constructing discontinuous solutions of system (1), we use an improved method of characteristics which makes it possible, in particular, to prove the existence of Lipschitz-continuous solutions of (1) under weaker restrictions on the initial functions.

Suppose that for system (2) the Cauchy problem is posed

$$r(0,x) = r^0(x); \qquad |x| \leq a, \tag{11}$$

where the vector function $r^0(x)$ satisfies the Lipschitz condition with respect to the variable x with bounded constant C (this Cauchy problem is equivalent to the Cauchy problem for system (1)). We shall call $r(t,x)$ a weak solution of the Cauchy problem (2), (11) if $r_k(t,x)$ are constant along the integral curves of the equations

$$\partial x_k/\partial t = \xi_k(r_1(t,x_k), r_2(t,x_k)) \qquad (k = 1, 2),$$

issuing from the segment $|x| \leq a$ of the initial axis.

Theorem 1. *For $0 \leq t \leq T(C)$ there exists a unique weak solution of problem (2), (11), satisfying the Lipschitz condition with respect to the variable x with constant $2C$. The theorem is valid under the single condition that the functions $\varphi_i(u)$ have second derivatives and $\xi_k(r)$ have first derivatives with respect to their arguments.*

We indicate here the method of successive approximations by means of which the solution of problem (2), (11) is constructed.

Suppose that an approximation

$$^{(n)}r(t,x) = \{\, ^{(n)}r_1(t,x);\ ^{(n)}r_2(t,x)\,\}$$

is known; moreover

$$^{(n)}r(0,x) = r^0(x).$$

Define

$$^{(n+1)}r(t,x)$$

as the solution of two Cauchy problems:

$$\frac{\partial\, ^{(n+1)}r_1}{\partial t} + \xi_1(^{(n+1)}r_1, {}^{(n)}r_2(t,x))\frac{\partial\, ^{(n+1)}r_1}{\partial x} = 0; \qquad ^{(n+1)}r_1(0,x) = r_1^0(x); \tag{12}$$

$$\frac{\partial\, ^{(n+1)}r_2}{\partial t} + \xi_2(^{(n)}r_1(t,x), {}^{(n+1)}r_2)\frac{\partial\, ^{(n+1)}r_2}{\partial x} = 0; \qquad ^{(n+1)}r_2(0,x) = r_2^0(x). \tag{13}$$

The functions

$$^{(n+1)}r_1(t,x) \quad \text{and} \quad ^{(n+1)}r_2(t,x)$$

are constant along the integral curves of the equations

$$\frac{d\,^{(n+1)}x_1}{dt} = \xi_1(^{(n+1)}r_1, {}^{(n)}r_2(t, {}^{(n+1)}x_1)); \qquad \frac{d\,^{(n+1)}x_2}{dt} = \xi_2(^{(n)}r_1(t, {}^{(n+1)}x_2), {}^{(n+1)}r_2). \tag{14}$$

If

$$^{(n)}r(t,x)$$

satisfies the Lipschitz condition, then through every point for $0 \leq t \leq T(C)$ there passes a unique integral curve of each of equations (14).

Lemma. *If*

$$^{(n)}r(t,x)$$

satisfies the Lipschitz condition with constant equal to $2C$*, then*

$$^{(n+1)}r(t,x)$$

for $0 \leq t \leq T(C)$ *also satisfies the Lipschitz condition with constant* $2C$*.*

The proof of the lemma and the definition of $T(C)$ are omitted here.

We must specify the first approximation satisfying the conditions of the lemma: set $\overset{(0)}{r}(t,x) = r^0(x)$. The proof of convergence of the method of successive approximations is based on the estimate

$$\left|\overset{(n+1)}{r}(t,x) - \overset{(n)}{r}(t,x)\right| \leq 4MCt \max_{\tau,\xi} \left|\overset{(n)}{r}(\tau,\xi) - \overset{(n-1)}{r}(\tau,\xi)\right|, \tag{15}$$

where $M > |\partial\xi_k/\partial r_j|$, from which follows the uniform convergence of the successive approximations to a weak solution $r(t,x)$. The uniqueness of the weak solution of problem (2), (11) also follows from estimate (15).

3. Construction of discontinuous solutions

The method described above for constructing Lipschitz-continuous solutions of system (1) can be applied up to the moment at which a singularity is formed in the solution. For problem (1), (11), such singularities are the occurrence of a jump in the solution, or the more frequent case—the unboundedness of the ratio $\Delta r_k/\Delta x$ as $\Delta x \to 0$. It is characteristic, however, that in all these cases the Riemann invariant $r_k(t,x)$ possessing the singularity is, in a neighborhood of the point containing the singularity, monotonically decreasing.

Therefore, let us consider for system (1) the Cauchy problem (11), assuming that the Riemann invariants $r_k^0(x)$ at the point $x = 0$ have an isolated singularity.

Fig. 2

Figure 2: Fig. 2

Theorem 2. *Let the initial data (11), given for* $|x| \le a$*, have at the point* $x = 0$ *an isolated singularity of one of the two types:*

1) $r^0(x)$ *is a discontinuous function; moreover* $r_k^0(-0) > r_k^0(+0)$*, and everywhere except the point* $x = 0$ *it satisfies the Lipschitz condition;*

2)

$$\frac{r_k^0(\Delta x) - r_k^0(0)}{\Delta x} \to -\infty \quad \text{as } \Delta x \to 0;$$

moreover $r_k^0(x)$ *decreases monotonically for* $|x| \le a$*. Then there exists* $T > 0$ such that for $0 \le t \le T$ there exists a unique generalized discontinuous solution of problem (1), (11).

Fig. 2

We shall not dwell on the definition of the quantity T and the proof of the theorem, restricting ourselves to a description of the method for constructing a generalized discontinuous solution of problem (1), (11) in the case where the initial function $r^0(x)$ is discontinuous.

By the method indicated above we solve the Cauchy problem (1), (11) separately on the intervals $-a \le x \le 0$ and $0 \le x \le a$. The solution $r(t, x)$ will be defined in the regions to the left of OL_1^- and to the right of OL_2^+ (Fig. 2). The construction differs somewhat in the cases

$$R_2(r_1^0(+0),\, r_1^0(-0),\, r_2^0(-0)) > r_2^0(+0)$$

and

$$R_2(r_1^0(+0),\, r_1^0(-0),\, r_2^0(-0)) \le r_2^0(+0).$$

We consider only the latter. We solve for system (2) the problem:

$$r_1\big|_{OL_2^+} = r_1(t, x)\big|_{OL_2^+}, \quad r_2(0, 0) = r_2^0(+0)+$$

$$+\alpha\left[R_2(r_1^0(+0),\, r_1^0(-0),\, r_2^0(-0)) - r_2^0(+0)\right], \quad 0 \le \alpha \le 1. \tag{16}$$

Such a problem reduces to that considered in § 2. Its solution will be defined in the zone $L_2^- O L_2^+$, with OL_2^- lying between OL_1^- and OL_2^+ (Fig. 2). The solution of this problem in the zone $L_2^- O L_2^+$ will also be denoted by $r(t,x)$. It is easy to establish that on the line OL_2^- $r_2(t,x)$ satisfies the Lipschitz condition.

Suppose that in the zone $L_2^- ONM$ a discontinuous function $\overset{(n)}{r}(t,x)$ is given, having a single line of discontinuity $OL_{D_1}^{(n)}$, and satisfying outside this line of discontinuity— the Lipschitz condition with constant C (to the left of $OL_{D_1}^{(n)}$, $r^{(n)}(t,x) = r(t,x)$).

We indicate a method for constructing the approximation $r^{(n+1)}(t,x)$. Let us solve the Cauchy problems*:

$$\frac{\partial \tilde{r}_1^{(n+1)}}{\partial t} + \xi_1(\tilde{r}_1^{(n+1)},\, r_2^{(n)}(t,x)) \frac{\partial \tilde{r}_1^{(n+1)}}{\partial x} = 0; \qquad \tilde{r}_1^{(n+1)}\Big|_{OL_2^-} = r_1(t,x)\big|_{OL_2^-}. \tag{17}$$

Let the solution of this problem be known in the zone $L_2^- O L_1^{(n+1)}$, containing within itself the line $OL_{D_1}^{(n)}$. It is established that $\tilde{r}^{(n+1)}(t,x)$ satisfies the Lipschitz condition with constant C. In the zone $L_1^{(n+1)} O L_1^-$ consider the ordinary differential equation:

$$\frac{dx_{D_1}^{(n+1)}}{dt} = D_1(\tilde{r}^{(n+1)}(t, x_{D_1}^{(n+1)}),\, r_1(t, x_D^{(n+1)}),\, r_2(t, x_D^{(n+1)})); \qquad x_{D_1}^{(n+1)}(0) = 0. \tag{18}$$

Since the right-hand side of (18) satisfies the Lipschitz condition and conditions (8) are fulfilled, there exists a unique integral curve of equation (18) issuing from the point $(0,0)$. We denote it by $OL_{D_1}^{(n+1)}$. After this we define $r_2^{(n+1)}(t,x)$:

$$\frac{\partial \tilde{r}_2^{(n+1)}}{\partial t} + \xi_2(\tilde{r}_1^{(n+1)}(t,x),\, \tilde{r}_2^{(n+1)}) \frac{\partial \tilde{r}_2^{(n+1)}}{\partial x} = 0;$$

$$\tilde{r}_2^{(n+1)}\Big|_{OL_{D_1}^{(n+1)}} = R_2(\tilde{r}_1^{(n+1)}(t,x),\, r_1(t,x),\, r_2(t,x))\Big|_{OL_{D_1}^{(n+1)}}. \tag{19}$$

The solution of problem (19) will satisfy the Lipschitz condition and is defined in the zone $L_{D_1}^{(n+1)} O L_2^-$. After this we define the approximation $r^{(n+1)}(t,x)$:

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* The magnitude T is limited by the possibility that a new singularity may appear.

$$r^{(n+1)}(t,x) = \begin{cases} r(t,x), & \text{to the left of } OL_{D_1}^{(n+1)}, \\ \tilde{r}^{(n+1)}(t,x), & \text{in the zone } L_{D_1}^{(n+1)} OL_2^-, \\ r(t,x), & \text{to the right of } OL_2^-. \end{cases} \tag{20}$$

The zeroth approximation is chosen so as to satisfy the assumptions made.

One proves the uniform convergence of the lines $OL_{D_1}^{(n)}$ to the discontinuity line OL_{D_1}, and the uniform convergence of the approximations $\tilde{r}^{(n)}(t,x)$ in any common part of the domains of definition of $\tilde{r}^{(n)}(t,x)$. Thus, there exists a limiting line OL_{D_1}, which is a line of strong discontinuity of $r(t,x)$. In view of the uniform convergence of the sequence $r^{(n)}(t,x)$, on the line OL_{D_1} the Hugoniot conditions (4) are fulfilled, and outside the discontinuity line the limit $r(t,x)$ is a weak solution of system (1).

The remaining possible cases are considered in an analogous way.

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* On the discontinuity lines of the coefficients of equation (17), the continuity condition is imposed on $r_1^{(n+1)}(t,x)$.

Note: Figure translations are in progress. See original paper for figures.

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