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Abstract

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ON THE CLASSICAL THEORY OF PERTURBATIONS AND THE PROBLEM OF STABILITY OF PLANETARY SYSTEMS

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§ 1. Perturbation theory makes it possible to predict the motion of the planets for many years ahead with all necessary accuracy. However, qualitative questions concerning the behavior of the system over an infinitely long interval of time–for example, the question of stability–perturbation theory has not been able to resolve. In this theory the motion of the planets is described by series of the form

$$\sum_{m,n} a_{mn} \cos[(m\omega_1 + n\omega_2)t + \varphi_{mn}].$$

In Laplace' s time it was considered inevitable that in higher approximations there would also appear "secular terms" of the form $at^\alpha \cos\omega t$ and bt^β. From this they tried to infer the instability of the Solar System (see ([1])). However, by the time of Poincaré' s works (([2,3]), see also ([4,5])) it had become clear that one can construct perturbation theory in such a way that the series of any approximation contain only terms of trigonometric type. On the other hand, it turned out that the indicated series diverge, so that the question of stability remained open.

The divergence of the series is connected with a kind of resonance–an approximate commensurability of frequencies. For example, the frequencies of Jupiter and Saturn, $\omega_1 \simeq 299'', 1$, $\omega_2 \simeq 120'', 5$, almost satisfy the relation $2\omega_1 = 5\omega_2$. The perturbation is expressed by means of the series

$$\sum_{m,n} \frac{a_{mn}}{m\omega_1 + n\omega_2} \cos[(m\omega_1 + n\omega_2)t + \varphi_{mn}].$$

Since the denominator $2\omega_1 - 5\omega_2$ is very small, a large long-period perturbation is observed (see ([4])). For most pairs ω_1, ω_2 the quantities $|m\omega_1 + n\omega_2|$ do not vanish and even exceed $K(|m|+|n|)^{-2}$ for some $K > 0$ and all integers $m, n > 0$ (see ([6])). This leads to the hypothesis, long discussed by astronomers, that for most initial conditions a planetary system is stable. To prove this hypothesis, however, was not possible because of difficulties of several types. Poincaré ([2,3]) proposed a number of model problems containing some of these difficulties separately.

The first nontrivial problems with small denominators were solved only in 1941 by Siegel ([8]). Siegel also gave (in certain cases; see ([9])) a rigorous proof of the fact, already known to Poincaré, that the approximations of perturbation theory can converge only in particular, exceptional cases.

An important step forward was made in 1954 by Kolmogorov, who applied a Newton-type method and constructed a convergent version of perturbation theory in the so-called nondegenerate case (see ([10,11])). The results of ([10]) have numerous applications; however, most problems of celestial mechanics belong precisely to the degenerate case.

After it became possible, by combining Newton' s method with the classical asymptotic methods, to overcome separately the difficulties connected with degeneracy of various kinds in model problems ([12,13,14]), the possibility opened up of applying the developed technique to the problem of the motion of the planets,

where all the difficulties occur together. In the present note we report results obtained along this path*.

§ 2. Let us consider, for simplicity, the planar three-body problem with masses M, m_1, m_2, where $m_1, m_2 \ll M$. Perturbation theory gives the following picture of the motion ([3, 4]). In the zero approximation the planets m_1, m_2 do not perturb one another and move along Keplerian ellipses with semimajor axes a_1, a_2 and eccentricities e_1, e_2. The directions of the major axes are determined by the angles ω_1, ω_2 (longitudes of perihelia). In the zero approximation a_k, e_k, ω_k are preserved throughout the motion. Let us consider the important case when m_k, e_k are small, $a_2 - a_1 > c$, and the planets move about M in the same direction.

In the first approximation, taking into account the mutual influence of the planets leads only to a small "trembling" of a_k, e_k, ω_k about constant values. In the second approximation, a slow but unbounded (secular) motion of the perihelia is found. This slow change of e_k, ω_k may be described as follows. We shall characterize the Keplerian ellipse by a vector directed along the major axis and proportional to the eccentricity. It turns out that, for each of the planets m_k, this vector is the sum of two uniformly rotating vectors ξ_{k1}, ξ_{k2}, whose angular velocities ν_1, ν_2 are small and the same for both planets. The motion of the planets along ellipses changing in this way will be called Lagrangian.

Our main result is that, if the masses and eccentricities of the planets are sufficiently small, then for the majority of initial conditions there exists a Lagrangian motion from which the true motion differs little over the entire infinite interval of time.

We shall regard the center of gravity of the bodies as fixed. Then the system has 4 degrees of freedom. Let $0 < c_1 < C_1 < c_2 < C_2 < \infty$ be constants.

The conditions $c_1 < a_1 < C_1$; $c_2 < a_2 < C_2$; $e_1, e_2 < \delta$ single out, in the 8-dimensional phase space, a domain G_δ. A point of G_δ uniquely determines the initial coordinates and velocities, and hence the entire motion. Let α_1, α_2 be constants and $m_1 = \mu\alpha_1 M$, $m_2 = \mu\alpha_2 M$.

Theorem 1. For any $\eta > 0$ there exists an $\varepsilon > 0$ such that, if $\delta, \mu < \varepsilon$, then the majority of points of the domain G_δ (the exceptions are points forming a set of measure less than $\eta \operatorname{mes} G_\delta$) move in such a way that: 1) the point forever remains in the domain G_δ; 2) it moves conditionally periodically, filling everywhere densely an analytic four-dimensional torus in G_δ; 3) it forever remains closer than η to the phase-space point representing some Lagrangian motion.

Remark 1. Analogous theorems on "metric stability" are valid in the planar n-body problem and in the spatial 3-body problem. Extending them to the spatial problem of $n > 3$ bodies requires some additional computations (connected with the elimination of the node).

Remark 2. The exceptional set in Theorem 1 extends to infinity, is connected, and is everywhere dense. Bearing in mind, on the one hand, these circumstances and, on the other, the known fact of the existence of "gaps" in the distribution of minor planets, one may suppose that the motion of the planets is topologically unstable.

§ 3. We outline the path of the proof of Theorem 1. As is known ([3]), the Hamiltonian function of the planar three-body problem has the form

$$F = F_0(\Lambda) + (\mu)F(\Lambda, \xi, \eta) + (\mu)\widetilde{F}(\Lambda, \lambda, \xi, \eta), \tag{1}$$

where $\Lambda = (\Lambda_1, \Lambda_2)$, $\lambda = (\lambda_1, \lambda_2)$; $\xi = (\xi_1, \xi_2)$, $\eta = (\eta_1, \eta_2)$ are canonically conjugate variables, corresponding as follows: Λ_k to the semimajor axes, the angles λ_k to the phases, and (ξ_k, η_k) to the vectors $(e_k \cos\omega_k,\ e_k \sin\omega_k)$. In formula (1) the bar denotes averaging over λ_1, λ_2, $(\mu)F = \mu F_1 + \mu^2 F_2 + \cdots$. The functions $F_0, \bar{F}, \widetilde{F}$ are analytic; $\widetilde{F}$ has period 2π in λ and mean value 0. For small ξ, η

* Some of them were announced in reports on 11 VII 1961 at the Fourth All-Union Mathematical Congress and on 27 XI 1961 at a conference on theoretical astronomy.

the functions $\overline{F}$ and $\widetilde{F}$ are expanded in a convergent Taylor series in ξ, η, with F containing only terms of even degree.

By a preliminary canonical transformation

$$\Lambda, \lambda, \xi, \eta \to \Lambda', \lambda', \xi', \eta'$$

("averaging over the fast variables," see § 4), one can reduce F in the greater part of the domain G_δ to the form

$$F' = F_0(\Lambda') + (\mu)\,\overline{F}(\Lambda', \xi', \eta') + F_2(\Lambda', \lambda', \xi', \eta'), \tag{2}$$

where F_2 is a perturbation of order μ^2.

The secular motion is determined by the Hamiltonian function $(\mu)\overline{F}(\xi', \eta')$, where the Λ'_k are regarded as parameters. The point $\xi' = \eta' = 0$ is an equilibrium position stable in the linear approximation. Introducing, following Birkhoff ([7]), new canonical variables r, φ in a neighborhood of zero, we reduce $\overline{F}$ to the form* $\overline{F} = \overline{F}_2(r) + R_3(r, \varphi)$, where $r = (r_1, r_2)$ are quantities of order e^2, $\varphi = (\varphi_1, \varphi_2)$ are angular variables; $(\mu)\overline{F}_2 = \nu_1 r_1 + \nu_2 r_2 + c_{11} r_1^2 + 2c_{12} r_1 r_2 + c_{22} r_2^2$, and R_3 begins with r^3, i.e. e^6. The canonical transformation

$$\Lambda', \lambda', \xi', \eta' \to \Lambda'', \lambda'', r, \varphi$$

brings F' to the form

$$F'' = F_0(\Lambda'') + (\mu)\,\overline{F}_2(\Lambda'', r) + F_2''(\Lambda'', \lambda'', r, \varphi), \tag{3}$$

where $F_2'' = F_2 + R_3$ is a perturbation of order $\mu^2 + \mu r^3$.

Conditionally periodic solutions of the equations with Hamiltonian function (3) are found by a convergent iterative method of Newtonian type. Here two difficulties arise. The first is connected with the limiting expression at $r = 0$ and is overcome in the same way as in ([12]). The second difficulty–the proper expression at $\mu = 0$–is connected with the presence of fast and slow motions. For $\mu = 0$ the Keplerian motion is described by two "fast" frequencies $\dot\lambda_1, \dot\lambda_2$, while for $\mu \neq 0$ in the Lagrangian motion two more "slow" frequencies ν_1, ν_2 (of order μ) appear (cf. ([13,14])).

Verification of the fulfillment of conditions (6) for the dependence of the frequencies on the momenta for the Hamiltonian function (3) was carried out by direct computations using the expansion of F in powers of e and a_1/a_2 ([15]).

§ 4. Let us formulate a generalization of the results of ([10]) to the case of a proper expression, when the Hamiltonian function has the form

$$H = H_0(p_1, \dots, p_k) + \varepsilon H_1(p_1, \dots, p_n, q_1, \dots, q_n) \quad (k < n), \tag{4}$$

is of period 2π in each of the variables $q_1, \dots, q_n$ and is analytic**, when p varies in some domain G and $|\operatorname{Im} q| < \rho$. For $\varepsilon = 0$ the canonical equations with Hamiltonian function (4) describe conditionally periodic motion**

$$\dot q_i = \omega_i \ (i \le k), \qquad \dot q_{k+1} = \dots = \dot p_n = 0$$

with frequencies

$$\omega_i = \partial H_0 / \partial p_i.$$

For small ε one may assume, neglecting the "trembling," that the slow change of $q_{k+1}, \dots, p_n$ with time is affected only by the mean value of H_1 over the fast variables*

$$\overline{H}_1(p; q_{k+1}, \dots, q_n) = (2\pi)^{-k} \oint H_1(p, q)\, dq_1 \dots dq_k.$$

We shall now assume**** that $\overline{H}_1$ does not depend on the phases of the slow motions. Then the Hamiltonian function (4) can be represented in the form

$$H = H_0(p_1, \dots, p_k) + \varepsilon \overline{H}_1(p_1, \dots, p_n) + \varepsilon \widetilde{H}_1(p_1, \dots, p_n, q_1, \dots, q_n). \tag{5}$$

* In the case of two planets $\overline{F}(\xi', \eta')$ can be reduced exactly to the form $\overline{F}(r)$. Our reasoning is applicable also in the case of $n > 2$ planets.

** *Remark added in proof.* In accordance with recent work of J. Moser, the existence of several hundred derivatives is sufficient.

*** This remark, going back to Gauss, constitutes the essence of the well-known "method of averaging" ([16]). What follows may be regarded as one of the variants of the justification of this method for an infinite interval of time.

**** This is precisely the situation in the problem of the motion of planets; see formula (3).

Theorem 2. Suppose that for $p \in G$ the conditions

$$\det \left| \frac{\partial^2 H_0}{\partial p_i \partial p_j} \right| \neq 0 \quad (i, j = 1, \dots, k); \qquad \det \left| \frac{\partial^2 H_1}{\partial p_i \partial p_j} \right| \neq 0 \quad (i, j = k+1, \dots, n). \tag{6}$$

are satisfied.

Denote by T the torus layer $\operatorname{Im} p = \operatorname{Im} q = 0$, $p \in G$, $q_i \in [0, 2\pi)$. Then for any $\eta > 0$ there exists $\varepsilon_0 > 0$ such that, if $|\varepsilon| < \varepsilon_0$, then in T there are analytic invariant n-dimensional tori carrying trajectories of conditionally periodic motions. These tori fill T up to a remainder of measure less than $\eta \operatorname{mes} T$.

Theorem 2 shows that, for small ε, for most initial conditions the motion of the system with Hamiltonian function (5) over an infinite time interval differs only slightly from the conditionally periodic n-frequency motion $\dot{q}_i = \partial \bar{H} / \partial p_i$ $(\bar{H} = \bar{H}_0 + \varepsilon \bar{H}_1)$, $\dot{p} = 0$ with suitable initial conditions.

The proof of Theorem 2 is analogous to the reasoning in [13], where the case $k = 1$ is considered. For $k > 1$, small denominators already appear at the first stage of the proof, in averaging over the fast variables. In order not to deal with an infinitely large number of resonances, it is convenient (cf., for example, [16]) to take into account in the perturbation $\widetilde{H}_1$ only harmonics up to order N; if $N \sim |\ln \varepsilon|$, the higher harmonics give a sum of order ε^2.

Let Ω be a domain in the space $\omega = (\omega_1, \dots, \omega_k)$ into which G is mapped under $p \to \partial H_0/\partial p$. Denote by Ω_{KN} the set of those ω for which $|(\omega, n)| > K|n|^{-s}$ $(s = k+1,\ |n| = |n_1| + \dots + |n_k|)$ for every integer nonzero vector n, $|n| < N$. By G_{KN} denote the preimage of Ω_{KN}, and by $G_{KN} - d$ the set of points belonging to G_{KN} together with a d-neighborhood.

In the proof of Theorem 2, a certain number $\delta > 0$ is chosen sufficiently small, and then $\varepsilon = \delta^T$ is chosen, where T is a sufficiently large constant depending only on the number of degrees of freedom n. The first step of the proof consists in establishing the following lemma.

Lemma. Under the conditions of Theorem 2, suppose $|\varepsilon H_1| < M$, $|\partial^2 H_0/\partial p_i \partial p_j| < \theta$. Suppose the numbers γ, δ satisfy the inequalities

$$2\gamma < \rho, \quad \delta < \min\left(K,\ 1/3,\ \gamma/4,\ 1/\theta,\ e^{2kn-2}(8k)^{-2k}\right), \quad M < \delta^{2k+7}.$$

Then in the domain $P \in G_{KN} - 2\delta$, $|\operatorname{Im} Q| < \rho - 2\gamma$ (where $N = \dfrac{1}{\gamma} \ln \dfrac{1}{M}$) there exists an analytic canonical one-to-one transformation $p, q \leftrightarrow P, Q$ bringing H to the form

$$H = H_0(P) + \varepsilon \bar{H}_1(P) + H_2(P, Q),$$

where

$$|P - p|,\ |Q - q| < M\delta^{-2k-5}; \qquad |H_2| < M^2 \delta^{-4k-10}.$$

Since H_2, in this way, has order ε^2 in the domain $G_{KN} - 2\delta$, while the magnitude of the components of the domain $G_{KN} - 2\delta$ is of order $|\ln \varepsilon^{-1}|$, the derivation of Theorem 2 from the lemma is carried out analogously to the proof of Theorem 2 in note [13].

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CITED LITERATURE

[1] C. Jacobi, *Lectures on Dynamics*, Moscow-Leningrad, 1936, p. 29.
[2] A. Poincaré, *On Curves Defined by Differential Equations*, Moscow-Leningrad, 1947.
[3] H. Poincaré, *Les méthodes nouvelles de la mécanique céleste*, Paris, 1, 1892; 2,

1893; 3, 1899.
[4] C. L. Charlier, *Die Mechanik des Himmels*, Leipzig, 1, 1902; 2, 1907.
[5] M. Born, *Lectures on Atomic Mechanics*, Kharkov–Kiev, 1934.
[6] A. Ya. Khinchin, *Continued Fractions*, Moscow, 1935.
[7] D. D. Birkhoff, *Dynamical Systems*, 1941.
[8] K. L. Siegel, *Lectures on Celestial Mechanics*, IL, 1959.
[9] K. L. Siegel, *Mathematics Collection*, 5, 2, 129 (1961).
[10] A. N. Kolmogorov, DAN, 98, No. 4, 527 (1954).
[11] International Mathematical Congress in Amsterdam, Moscow, 1961.
[12] V. I. Arnold, DAN, 137, No. 2, 255 (1961).
[13] V. I. Arnold, DAN, 138, No. 1, 13 (1961).
[14] V. I. Arnold, DAN, 142, No. 4, 758 (1962).
[15] U.-J. Le Verrier, *Ann. de l' Observatoire Impérial de Paris*, 1 (1855).
[16] N. N. Bogolyubov, Yu. A. Mitropolsky, *Asymptotic Methods in the Theory of Nonlinear Oscillations*, 1958.

Note: Figure translations are in progress. See original paper for figures.