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Fig. 1

Figure 1: Fig. 1

Abstract

Full Text

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ON THE STABILITY OF A THICK PLASTICALLY DEFORMABLE PLATE

(Presented by Academician A. Yu. Ishlinskii, 26 IV 1962)

According to the idea of L. S. Leibenzon (^{1}) (see also (^{2})), the stability of thick-walled structures should be studied on the basis of the general equations of the elastic-plastic state. A number of problems in such a formulation have been considered in papers ($^{3-6}$) and others. In paper (^{7}), the loss of stability of a thick simply supported plate under a compressive load was considered in an analogous way, using the relations of the theory of elastic-plastic deformations. In the present paper the same problem is solved on the basis of the equations of the flow theory for an anisotropically hardening material ($^{8-10}$).

Fig. 1

Consider a thick rectangular simply supported plate, uniformly compressed along the z-axis by a load of intensity p (see Fig. 1). For simplicity we shall assume the material to be incompressible.

We take the flow function in the form

$$[\sigma_x - \sigma_y - C(\varepsilon_x^p - \varepsilon_y^p)]^2 + [\sigma_y - \sigma_z - C(\varepsilon_y^p - \varepsilon_z^p)]^2 +$$

$$+[\sigma_z - \sigma_x - C(\varepsilon_z^p - \varepsilon_x^p)]^2 +$$

$$+6[(\tau_{xy} - C\varepsilon_{xy}^p)^2 + (\tau_{xz} - C\varepsilon_{xz}^p)^2 + (\tau_{yz} - C\varepsilon_{yz}^p)^2] = 2f^2(\Gamma), \qquad (1)$$

$$6\Gamma = (\varepsilon_x^p - \varepsilon_y^p)^2 + (\varepsilon_y^p - \varepsilon_z^p)^2 + (\varepsilon_z^p - \varepsilon_x^p)^2 + 6(\varepsilon_{xy}^{p\,2} + \varepsilon_{xz}^{p\,2} + \varepsilon_{yz}^{p\,2}),$$

$$C = C(\Gamma).$$

Here and below the superscript p is assigned to the components of plastic strain; the components of elastic strain will be assigned the superscript e.

For $f = k$, the flow surface during deformation is displaced in stress space as a rigid whole. For $C = 0$ there is isotropic expansion of the yield surface. For $f \neq 0$ and $C \neq 0$, both of the indicated effects occur.

We shall seek the solution in the form

$$\sigma_{ij} = \sigma_{ij}^0 + \sigma'_{ij}; \qquad \varepsilon_{ij} = \varepsilon_{ij}^0 + \varepsilon'_{ij}; \qquad u = u^0 + u'; \dots, \tag{2}$$

where the superscript zero is assigned to the components of the unperturbed state, and the prime superscript to those of the perturbed state; u, v, w are displacements along the axes x, y, z, respectively.

The total strain is composed of elastic and plastic parts: $\varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^p$; the components of elastic strain are related to the stresses by Hooke' s law. The relation between increments of plastic strain and stress—

we obtain from the associated flow law

$$d\varepsilon_{ij}^p = d\lambda \frac{\partial f}{\partial \sigma_{ij}}, \qquad d\lambda \geqslant 0. \tag{3}$$

We shall assume that the plastic potential f coincides with the yield function (1).

For uniaxial compression in the unperturbed state, for $z = \pm l$, $y = \pm b$, we obtain

$$\sigma_z^0 \neq 0, \qquad \sigma_x^0 = \sigma_y^0 = \tau_{xy}^0 = \tau_{xz}^0 = \tau_{yz}^0 = 0. \tag{4}$$

From Hooke' s law we find

$$\varepsilon_{xy}^{e0} = \varepsilon_{xz}^{e0} = \varepsilon_{yz}^{e0} = 0; \qquad \varepsilon_x^{e0} = \varepsilon_y^{e0} = -{}^1/{}_2\varepsilon_z^{e0}; \qquad \varepsilon_z^{e0} = \sigma_z^0/E. \tag{5}$$

Substituting relations (2) into (1), (3) and linearizing them, taking into account (4) and (5), for the unperturbed state we shall have

$$\varepsilon_x^{p0} = \varepsilon_y^{p0} = -{}^1/{}_2\varepsilon_z^{p0}; \qquad \varepsilon_{xy}^{p0} = \varepsilon_{xz}^{p0} = \varepsilon_{yz}^{p0} = 0; \tag{6}$$

$$\Gamma^0 = {}^3/{}_4\,(\varepsilon_z^{p0})^2; \qquad \sigma_z^0 = f(\Gamma^0) + {}^3/{}_2C(\Gamma^0)\varepsilon_z^{p0}.$$

For a given modulus of elasticity E, the character of hardening is determined by the dependence $\sigma \sim \varepsilon^p$. Let us assume that a functional relation is prescribed

$$\sigma_z^0 = \Phi(\varepsilon_z^{p0}). \tag{7}$$

Then under proportional loading

$$f(\Gamma) = \Phi(2\sqrt{\Gamma/3}) - \sqrt{3\Gamma}\, C(\Gamma). \tag{8}$$

Here the superscript zero index is omitted.

The increments of the strain components at loss of stability are nothing other than the components of the perturbation; therefore,

$$d\varepsilon_{ij} = \varepsilon'_{ij}, \qquad d\varepsilon^p_{ij} = \varepsilon^{p'}_{ij}. \tag{9}$$

For the elastic components of the perturbed state, from Hooke' s law we shall have

$$\varepsilon_x^{e'} = \frac{1}{2E}\,(2\sigma'_x - \sigma'_y - \sigma'_z), \qquad \varepsilon_{xy}^{e'} = \frac{1}{G}\,\tau'_{xy}, \qquad (x, y, z);$$

$$\varepsilon'_{xy} = \frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x}, \dots, \tag{10}$$

where (x, y, z) means that the missing expressions are obtained from the indicated ones by cyclic permutation of the indices.

From relations (1) and (3) we obtain

$$\Gamma' = {}^3/_2\varepsilon_z^{p0}\varepsilon_z^{p'}; \qquad \varepsilon_x^{p'} = \varepsilon_y^{p'} = -{}^1/_2\varepsilon_z^{p'}; \qquad \varepsilon_{xy}^{p'} = \varepsilon_{xz}^{p'} = \varepsilon_{yz}^{p'} = 0;$$

$$2\sigma'_z - \sigma'_x - \sigma'_y = 3\left[\varepsilon_z^{p0}\left(\frac{df}{d\Gamma}\right)^0 + C(\Gamma^0) + {}^3/_2\left(\frac{dC}{d\Gamma}\right)^0 (\varepsilon_z^{p0})^2\right]\varepsilon_z^{p'}; \tag{11}$$

$$\left(\frac{d}{d\Gamma}\right)^0 = \left.\frac{d}{d\Gamma}\right|_{\Gamma=\Gamma^0},$$

whence it follows that the total components of the shear-strain increments are elastic:

$$\varepsilon'_{xy} = \varepsilon_{xy}^{e'}, \qquad (x, y, z). \tag{12}$$

The total strains of the perturbed state will be written in the form

$$\frac{\partial u'}{\partial x} = \varepsilon_x^{e'} + \varepsilon_x^{p'} = \frac{1}{2E}(2\sigma_x' - \sigma_y' - \sigma_z') - \frac{1}{2A_0}(2\sigma_z' - \sigma_x' - \sigma_y');$$

$$\frac{\partial v'}{\partial y} = \frac{1}{2E}(2\sigma_y' - \sigma_z' - \sigma_x') - \frac{1}{2A_0}(2\sigma_z' - \sigma_y' - \sigma_x'); \qquad (13)$$

$$\frac{\partial w'}{\partial z} = \left(\frac{1}{2E} + \frac{1}{A_0}\right)(2\sigma_z' - \sigma_x' - \sigma_y'),$$

where

$$A_0 = 3\left[\varepsilon_z^{p0}\left(\frac{df}{d\Gamma}\right)^0 + C(\Gamma^0) + {}^3/2\left(\frac{dC}{d\Gamma}\right)^0 (\varepsilon_z^{p0})^2\right], \qquad (14)$$

or, taking (8) into account,

$$A_0 = 2(d\Phi/d\Gamma)^0.$$

The three relations (13) reduce to two

$$\sigma_x' - \sigma_y' = \frac{2E}{3}\left(\frac{\partial u'}{\partial x} - \frac{\partial v'}{\partial y}\right); \qquad 2\sigma_z' - \sigma_y' - \sigma_x' = \frac{2EA_0}{2E + A_0}\frac{\partial w'}{\partial z}. \qquad (15)$$

The equilibrium equations

$$\frac{\partial \sigma_x'}{\partial x} + \frac{\partial \tau_{xy}'}{\partial y} - \frac{\partial \tau_{xz}'}{\partial z} = 0 \quad (x, y, z) \qquad (16)$$

with the expressions (10) taken into account are written in the form

$$\frac{\partial \sigma_x'}{\partial x} = -G\left[\Delta u' - 2\frac{\partial^2 u'}{\partial x^2}\right]; \qquad \frac{\partial \sigma_y'}{\partial y} = -G\left[\Delta v' - 2\frac{\partial^2 v'}{\partial y^2}\right];$$

$$\frac{\partial \sigma_z'}{\partial z} = -G\left[\Delta w' - 2\frac{\partial^2 w'}{\partial z^2}\right]; \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \qquad (17)$$

Differentiating expressions (15) with respect to x, y, z and comparing them with the analogous expressions obtained from (17), using the incompressibility condition and the condition that $E = 3G$, we obtain

$$\partial \Delta u'/\partial y - \partial \Delta v'/\partial x = 0,$$

$$(6+D_0)\frac{\partial^4}{\partial x \partial y \partial z^2}\left[\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right] - 2\frac{\partial^2}{\partial x \partial y}\left[\frac{\partial \Delta u'}{\partial x} + \frac{\partial \Delta v'}{\partial y}\right] - \frac{\partial^2}{\partial z^2}\left[\frac{\partial \Delta u'}{\partial y} + \frac{\partial \Delta v'}{\partial x}\right] = 0, \tag{18}$$

where $D_0 = -6A_0/(2E + A_0)$.

It is easy to see that, for A_0 tending to infinity, the case of elastic loss of stability will occur.

We seek the solution of system (18) in the form

$$u' = \varphi(x)\cos(ny)\cos(mz), \qquad v' = \psi(x)\sin(ny)\cos(mz). \tag{19}$$

Since, for a freely supported plate, $u' = 0$ when $y = \pm b$, $z = \pm l$, expressions (19) give

$$nb = \pm\pi/2 + i\pi, \qquad ml = \pm\pi/2 + j\pi, \qquad i, j = 0, 1, 2, \ldots$$

Proceeding in the same way as in paper (7), we find the transcendental equation for determining the critical value σ_z^{0*}

$$\sigma_z^{0*} = E\{\lambda(\gamma^2 + \alpha^2)[m^2(\lambda^2 + \alpha^2) + 2n^2(\lambda^2 - \alpha^2)]\,\mathrm{th}(\gamma a)$$

$$-\gamma(\lambda^2\alpha^2)[m^2(\gamma^2 + \alpha^2) + 2n^2(\gamma^2 - \alpha^2)]\,\mathrm{th}(\lambda a) + 4n^2\alpha\lambda\gamma\ \mathrm{th}(\alpha a)\}$$

$$: 3m^4\{\lambda(\gamma^2 + \alpha^2)\,\mathrm{th}(\gamma a) - \gamma(\lambda^2 + \alpha^2)\,\mathrm{th}(\lambda a)\}, \tag{20}$$

where

$$\alpha^2 = m^2 + n^2, \quad \lambda = \frac{1}{2}\{4n^2 + m^2[4 - R + R\sqrt{R - 8}]\}^{1/2},$$

$$\gamma = \frac{1}{2}\{4n^2 + m^2[4 - R - R\sqrt{R - 8}]\}^{1/2}, \qquad R = 6 + D_0.$$

For a plate of small thickness we shall have

$$\sigma_z^{0*} = -\frac{Ea^2}{9m^4}\{m^2(\lambda^2 + \alpha^2)(\gamma^2 + \alpha^2) + 2n^2(\gamma^2 - \alpha^2)(\lambda^2 - \alpha^2)\}. \tag{21}$$

In the case of elastic deformations, as in [7], for the critical load we obtain the formulas of the Kirchhoff-Love theory [11], which are limiting cases of the relations found.

Formulas (20) and (21) coincide, up to a factor, with the corresponding expressions for loss of stability of a plate obeying the relations of the theory of small elastic-plastic deformations [7].

One may write

$$\frac{(\sigma_z^{0*})_1}{E} = \frac{(\sigma_z^{0*})_2}{E_c}, \tag{22}$$

where $(\sigma_z^{0*})_1$ is the critical stress according to the flow theory, $(\sigma_z^{0*})_2$ is the critical stress according to the theory of small elastic-plastic deformations, and E_c is the secant modulus.

Expression (22) is valid for both thin and thick plates. The fact that, for materials obeying the flow theory, the unloading modulus coincides with the modulus of elasticity has long been known (see, for example, [12]). In the present case it has been shown that this circumstance also determines the character of loss of stability in the case of thick-walled plates.

Better agreement with experimental data may be achieved by introducing corner (conical) points on the yield surface.

It should be noted that loss of stability is determined only by the form of the curve $\sigma_z^0 \sim \varepsilon_z^0$, while the character of hardening (isotropic or anisotropic) has no effect whatever on the values of the critical loads.

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Note: Figure translations are in progress. See original paper for figures.

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