CYBERNETICS AND CONTROL THEORY

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Abstract

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ON ESTIMATES OF THE NUMBER OF PLANAR GRAPHS

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In many problems of the synthesis of logical circuits, in particular contact circuits, topological problems arise that are connected with the properties of networks and graphs. One such problem is to determine the cardinality of the set of networks of one kind or another used in the realization of logical functions. In the present note planar graphs and their planar realizations are considered. It is shown that the number G_n of connected nonisomorphic planar graphs with n edges is estimated as follows:

$$C_1^n < G_n < C_2^n, \tag{1}$$

where C_1 and C_2 are certain constants. These inequalities make it possible to obtain an asymptotic expression for the complexity of realizing functions of an algebra of logic by planar circuits. Questions connected with the number of planar realizations of a graph and certain others are also considered.

§ 1. It is known that every graph has a realization in three-dimensional space, consisting of a system of points corresponding to the vertices and of pairwise nonintersecting (except at their ends) arcs joining incident vertices. In this case different realizations of one graph are, obviously, isomorphic. A graph is called **planar** if it has a realization situated in the plane (a planar realization)*.

Definition. Planar realizations of a planar graph are called **maps**. Two maps are called **isotopic** if their edges and regions (contours) can be numbered so that regions of the same name are bounded by edges of the same name and, at the corresponding vertices, the edges follow (for example, clockwise) in the same order. Isotopic maps can be transformed into one another without leaving the plane.

§ 2. A **tree** is a connected graph without cycles. A tree with a distinguished vertex is called **rooted**, and this vertex is the **root**. Every tree is a planar graph. A map realizing it will be called a **plane tree**.

Definition. A **river** is a plane rooted tree with an arbitrarily distinguished edge incident with the root (it is called the **initial** edge). Two rivers will be called

Fig. 1

Figure 1: Fig. 1

isotopic if there exists an isotopy of the corresponding plane trees preserving the correspondence between the roots and the initial edges.

We shall now define a certain completely unambiguous traversal of a river. Namely, we shall move, starting from the root, along the initial edge, leaving it on our right, and then continue the path as if through a labyrinth, touching the wall with the right hand. (This traversal can also be imagined as going around a fence which, in the plane, represents the given river.) We shall encode rivers by strings of zeros and ones, writing, during the traversal of a river, successively 0 or 1 depending on whether we pass each edge for the first or the second time. Thus, for example, the code of the river shown in Fig. 1a is

* Planar graphs defined in this way satisfy the completely discrete definition (1).

the sequence 000100111010110100100111101. As is easy to see, the code of a river is determined uniquely and is a sequence of n zeros and n ones, where n is the number of edges of the river.

Theorem 1. In order that a sequence of n zeros and n ones be the code of some river, it is necessary and sufficient that, for every m, among the first m symbols of the sequence there be no fewer than half zeros.

Proof. The number R_n of nonisotopic rivers with n edges is equal to the number of sequences of n zeros and n ones satisfying the condition of the theorem. It can be shown*, that

$$R_n = \frac{1}{n+1} C_{2n}^{n}. \tag{2}$$

Fig. 1

§ 3. We shall call an (m,k)-**rosette** a graph containing k vertices $a_1, a_2, \dots, a_k$ and $(m+k)$ edges $a_1, a_2, \dots, a_k$ (marked edges) and $b_1, b_2, \dots, b_m$, connecting the vertices of the graph subject to the following conditions: 1) the edge a_i has endpoints a_i and $a_{i+1 \pmod k}$ (thus the edges $\{a_i\}$ form a polygon, or a cycle of length k, $a_1 a_1 a_2 a_2 a_3 \dots a_{k a_{ka}} 1$, hereafter called the k-cycle); 2) if two edges b_i and b_j respectively connect (not necessarily distinct) vertices a_{i_1} with a_{i_2} and a_{j_1} with a_{j_2}, then on the k-cycle these vertices are arranged either in the order $a_{i_1} a_{i_1} a_{j_2} a_{i_2}$, or in the order $a_{i_1} a_{i_2} a_{j_1} a_{j_2}$, or, finally, in the order $a_{i_1} a_{i_2} a_{j_2} a_{j_1}$ (meaning the order in the direction $a_1 a_2 \dots a_{ka} 1$).

Fig. 2

Fig. 2

Figure 2: Fig. 2

Fig. 3

Figure 3: Fig. 3

It can be shown that an (m, k)-rosette graph is planar and that there exists a planar realization of it which consists of a marked k-cycle, traversed in the direction $a_1 a_2 \dots a_{ka} 1$ clockwise, and also has m unmarked edges joining vertices of the cycle externally (i.e., not passing inside the k-cycle, which therefore may be called the k-contour). Loops and multiple edges are allowed. Such a map will be called an (m, k)-rosette, or simply a rosette if it is clear from the context what m and k are or if this is immaterial (Fig. 2a shows an $(11, 10)$-rosette).

On the outer boundary of a rosette there is at least one vertex of the cycle (α in Fig. 2a). Choose such an exterior vertex and draw through it some line connecting the exterior region of the rosette with the interior of the k-contour. We now "split" the chosen vertex into two new vertices (a_1 and a_2 in Fig. 2b) so that each edge incident with the chosen vertex is now incident with one or the other new vertex, depending on the side from which it approaches the chosen line. As a result—

* See, for example, ([2]), pp. 33-35, Example 5.

** From equality (2) it follows that $R_n \sim \frac{1}{\sqrt{\pi}} \cdot 4^n n^{-3/2}$. Pólya ([3]) showed that the number D_n of nonisomorphic rooted trees with n edges is asymptotically equal to

$$D_n \sim AC^n n^{-3/2}, \tag{a}$$

where A and C are certain constants, with $e \leq C \leq 4$. Equality (2) shows that the lowering of the upper estimate for D_n, namely of the constant C in equality (a), can be obtained only by taking into account the ambiguity of the coding corresponding to the ambiguous embedding of a tree in the plane (see, in this connection, § 5 of the present note). The uniqueness of the embedding of terminal edges allowed E. D. Sotskov to show that $C \leq 3.64$.

as a result we obtain an unfolded rosette: a chain of marked edges, on one side of which there are edges connecting the vertices of the chain.

Let $N_1^{m,k}$ be the number of nonisotopic (m, k)-rosettes; $N^{m,k}$ the number of nonisotopic unfolded (m, k)-rosettes; $N_2^{m,k}$ the number of nonisotopic (m, k)-rosettes with a marked outer vertex and a chosen line connecting the interior of the k-contour with the outer region. The inequalities

Fig. 4

Figure 4: Fig. 4

Fig. 3

hold:

$$N_1^{m,k} < N^{m,k} = N_2^{m,k} < (2m+k)N_1^{m,k}. \tag{3}$$

Lemma 1.

$$N^{m,k} = R_m C_{2m+k}^k = \frac{1}{m+1} C_{2m}^m C_{2m+k}^k. \tag{4}$$

Proof. There exists a one-to-one correspondence between an unfolded (m,k)-rosette and a pair of figures (Fig. 3), of which the first is a chain of length k, with $2m$ terminal edges attached on one side in such a way that at each vertex of the chain there are incident just as many edges as are incident with the corresponding vertex of the unfolded (m,k)-rosette (such a figure we shall call a "comb"); the second figure is a river with m edges, dual to the unmarked edges of the unfolded rosette (the construction is clear from Fig. 3a; as the initial edge one should take, for example, the first from the right among the edges incident with the root). The comb obtained from an unfolded (m,k)-rosette is encoded one-to-one by an ordered decomposition of the number k into $2m+1$ integral nonnegative summands. Therefore the number of distinct combs of this kind is equal to C_{2m+k}^k, whence (4) follows.

Fig. 4

§ 4. Every connected graph has as a subgraph a tree containing all its vertices, called a spanning tree of the graph[4]. A map has a plane tree as its spanning tree. Having chosen in it a root and an initial edge, we turn it into a river; here, as the root we choose a vertex on the outer boundary of the map.

Let K_n be the number of nonisotopic connected maps with n edges.

Theorem 2. $K_n < 4 \cdot 16^n$.

Proof follows from the existence of a one-to-one correspondence between maps with spanning trees chosen in them—rivers (with root on the outer boundary) —and pairs of figures (see Fig. 4) consisting of a rosette with a chosen outer vertex and a river. Indeed, the traversal of the spanning tree is a cycle in which each edge of the spanning tree occurs twice (see Figs. 1b, 4b). The remaining edges of the map are attached to this cycle externally; that is, from a map with n edges and k vertices one obtains an $(n-k+1,\, 2k-2)$-rosette with a marked outer vertex corresponding to the root of the spanning tree. Hence, if by $K'_{n,k}$

we denote the number of distinct connected maps with n edges, k vertices, and a chosen spanning tree—a river—then there holds

(by Lemma 1) $K'_{n,k} < R_{k-1} N^{n-k+1,\, 2k-2} < \frac{2}{n} \cdot 16^n$. Since

$$K'_n = \sum_{k=1}^{n+1} K'_{n,k} < \frac{2(n+1)}{n} \cdot 16^n \leqslant 4 \cdot 16^n,$$

the theorem is proved, since $K_n < K'_n$.

Corollary. From (a) and $G_n \leqslant K_n < C_2 \cdot 16^n$, (1) follows.

Let $R^{m,k}$ be the number of nonisotopic maps in which one can choose a k-contour so that they thereby become (m,k)-rosettes.

Lemma 2. $R^{m,k} \geqslant \frac{1}{n} N^{m,k}$, where $n = m + k$.

Theorem 2. $K_n > \frac{C_3}{n^3} Z^n$, where

$$Z = \frac{(2+\sqrt{2})^{(2+\sqrt{2})/2}}{(2-\sqrt{2})^{(2-\sqrt{2})/2} (\sqrt{2})^{\sqrt{2}}} \simeq 5.83\ldots,$$

Proof. For $m + k = n$, from Lemma 2 we have

$$K_n > \max_{m+k=n} R^{m,k} \geqslant \frac{1}{n} \max_m \left(\frac{1}{m+1} C_{2m}^m C_{n+m}^{n-m} \right).$$

The maximum of the expression enclosed in parentheses is attained for $m = \frac{\sqrt{2}}{2} n + O(1)$, whence the assertion of the theorem follows (C_3 is a constant).

§ 5. Let U_n (respectively V_n) be the maximum number of nonisotopic planar realizations of a graph (respectively of a tree) with n edges, i.e., nonisotopic embeddings in the plane.

Theorem 4.

$$(4\gamma_1(n))^n \geqslant V_n \geqslant (4\gamma_2(n))^n,$$

where

$$\lg \gamma_1(n) = -\frac{3 \lg n}{2n} + O\left(\frac{1}{n}\right); \qquad \lg \gamma_2(n) = -\frac{2 \lg \lg n}{\lg n} + O\left(\frac{1}{\lg n}\right);$$

$\lg x$ denotes $\log_4 x$.

Proof. As the upper estimate, R_n is taken. The lower estimate is obtained effectively: the tree formed by joining (at the root) all possible trees with p edges (p is the largest integer for which $n \geqslant \frac{t}{p+1} C_{2p}^{p}$) has at least $\left(\frac{n}{p} - 1\right)!$ nonisotopic embeddings.

Lemma 3. An (m, k)-rosette graph without loops has no more than $2(m+1)$ nonisotopic embeddings in the form of an (m, k)-rosette.

Lemma 4. An (m, k)-rosette graph has no more than $(m+1) \cdot 4^m$ nonisotopic embeddings in the form of an (m, k)-rosette.

Theorem 5.

$$n \cdot 4^n \gtrsim U_n \gtrsim \frac{1}{16\sqrt{\pi n}} \cdot 4^n; \qquad \alpha \gtrsim \beta \text{ means } \lim \alpha/\beta \geqslant 1.$$

Proof. The upper estimate follows from Lemma 4, Theorem 4, and the fact that an embedding of a planar graph is composed of an embedding of its core and a subsequent embedding of the remaining edges, which reduces to the embedding of a rosette graph. The lower estimate is obtained effectively: an $(n, 2)$-rosette graph, all unmarked edges of which are loops incident to one vertex, has C_{2n-4}^{n-2} nonisotopic embeddings.

§ 6. Let $L_{\text{pl}}(k)$ be the minimum number of contacts sufficient to realize any function of the algebra of logic in k variables by a planar circuit (cf. (5)). From (1) it follows that, for the number S_n of nonisomorphic connected two-terminal planar networks, the relation $C_4^n \gtrsim S_n$ holds, where C_4 is a constant, and hence from (6) $L_{\text{pl}}(k) \sim 2^k / \log_2 k$. This shows that the use of planar circuits does not change the asymptotic complexity of circuits in comparison with the use of series-parallel circuits.

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