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Abstract

Full Text

MATHEMATICS

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SOME RESULTS ON PERFECT MEASURES

(Presented by Academician A. N. Kolmogorov, November 18, 1961)

Let us recall the definition of perfect measures given by B. V. Gnedenko and A. N. Kolmogorov in ⁽¹⁾. A measure μ on a σ -algebra S of subsets of a space X is called **perfect** if, for every $\varepsilon > 0$, every real S -measurable function f , and every set $E \subset R$ with measurable f -preimage, there exists an open set $G \supset E$ such that

$$\mu(f^{-1}(E)) > \mu(f^{-1}(G)) - \varepsilon^*.$$

1. Perfect measures are closely connected with Lebesgue measures in the sense of V. A. Rokhlin ⁽²⁾. Namely, the following holds:

Theorem 1. *A complete measure is perfect if and only if its factor-measure with respect to any measurable partition ⁽²⁾ is a Lebesgue measure.*

The proof of this theorem is based on the following lemma. Let (X, S, μ) be a space with a perfect measure, ξ a measurable partition of X generated by a sequence $\{E_i\}$, and S_1 the σ -algebra generated by this sequence.

Lemma. *For any set $E \in S$ that is a union of elements of the partition ξ , there exist two sets $E_1 \in S_1$ and $E_2 \in S_1$ such that $E_1 \subset E \subset E_2$ and $\mu(E_1) = \mu(E_2)$.*

It follows at once from the lemma that if, in a complete perfect space, there is a countable class of measurable sets separating points, then this space is a Lebesgue space.

2. Let (X, S, μ) be a space with a perfect measure, $Y \subset X$, and $\mu^*(Y) = \mu(X)$ (here μ^* is the outer measure generated by the measure μ). Further, let μ_Y be the measure on the σ -algebra SY of sets of the form $E \cap Y$, $E \in S$ (we shall also use this notation below), induced by the measure μ (see ⁽⁴⁾, p. 78). The theorem on subspaces of a Lebesgue space (see ⁽²⁾, p. 117) can be generalized to perfect spaces as follows:

Theorem 2. *The measure μ_Y is perfect if and only if, for every measurable partition ξ of the space X , the canonical image Y in the factor-space $X | \xi$ belongs to S^ξ , where S^ξ is the domain of definition of the completion of the factor-measure μ^ξ .*

3. It can be shown that, for the perfection of every measure on a measurable space $(X, \mathfrak{B}X)$, where $X \subset \mathbb{R}$ and \mathfrak{B} is the σ -algebra of Borel sets of the line, it is necessary and sufficient that X be absolutely measurable. With the aid of this assertion it is easy to give an answer to the question posed by Blackwell in ⁽³⁾: suppose that on a measurable space (X, S) , whose σ -algebra S is generated by a countable number of sets, every measure is perfect; is it true that $f(X)$, for every real S -measurable function, is an analytic set? The answer is negative: it suffices to consider the space $(X, \mathfrak{B}X)$ and the function $f(x) = x$, where X is some absolutely measurable nonanalytic set, for example the complement of an analytic non-Borel set.

* In contrast to (1), however, we do not require μ to be complete.

4. Let μ be a measure on some σ -algebra S of subsets of a topological space X . Recall that μ is called regular if, for every $A \in S$,

$$\mu(A) = \sup_{S \ni F \subset A} \mu(F),$$

where F denotes a closed set; μ is called tight if, for every $\varepsilon > 0$, there exists a compact set $K \subset X$ such that

$$\mu^*(K) > 1 - \varepsilon;$$

and μ is called τ -smooth if

$$P(F_\alpha) \rightarrow 0$$

for every directed decreasing system $\{F_\alpha\}$ of closed sets such that $F_\alpha \downarrow \emptyset$ and $F_\alpha \in S$. Every tight measure is τ -smooth.

In the case of an arbitrary topological space, perfection is only weakly connected with smoothness.

Theorem 3. *In every topological space, regular tight distributions are perfect.*

One can give, however, examples of (Borel and Baire) measures: a) regular, τ -smooth, but not tight; b) regular, perfect, not τ -smooth; c) regular, perfect, τ -smooth, not tight.

In topological spaces with a countable base and in a broad class of metric spaces, perfection and smoothness are much more closely connected.

Theorem 4. *A Borel measure in a topological space with a countable base is tight if and only if it is perfect.*

Corollary 1. A Borel measure in a metric space is tight if and only if it is perfect and τ -smooth.

Corollary 2. If in a metric space X there does not exist a system of pairwise disjoint nonempty open sets of cardinality greater than the cardinality of the continuum, then a Borel measure in X is tight if and only if it is perfect.

5. Finally, let us note that Theorem 1 in ⁽¹⁾, p. 23, can be generalized as follows:

Theorem 5. *In order that a measure defined on a σ -algebra of subsets of a topological space with a countable base, containing all open sets, be perfect, it is necessary and sufficient that the measure of any set be equal to the supremum of the measures of compact sets lying inside this set.*

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- ³ D. Blackwell, Proc. Third Berkeley Symposium on Math. Statist. and Prob., **2**, 1 (1956).
- ⁴ P. Halmos, *Measure Theory*, IL, 1953.

Note: Figure translations are in progress. See original paper for figures.

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