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Abstract

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ON A VERTEX-EDGE FUNCTION OF GRAPHS

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A systematic study of recurrent functions of graphs was begun by A. A. Zykov ([1]-[4]). We shall consider one of such functions, belonging to the class of vertex-edge functions ([3]), and in doing so shall use the definitions and notation of ([1]-[4]).

1. Let L be an arbitrary nonempty graph whose edges are ordered in some way; let ab be its first edge. Let L_α be the graph obtained from L by deleting the edge ab without deleting vertices; let L_μ be the graph obtained from L by deleting the edge ab and replacing the vertices a and b by a single vertex $\bar{a}$, adjacent to those of the remaining vertices of L which were adjacent to one and only one of the vertices a, b (the order of the edges in the graphs L_α, L_μ is induced by the order of the edges in the graph L).

Fig. 1

Further, let K be a ring with generators $\alpha, \mu, 1$, and let $\Phi(L)$ be a function of graphs taking values in K and satisfying the conditions

$$\Phi(L) = \alpha\Phi(L_\alpha) + \mu\Phi(L_\mu) + 1; \tag{1}$$

$$\Phi(E_n) = 0, \qquad n = 0, 1, 2, \dots, \tag{2}$$

where E_n is the empty n-vertex graph.

In order that the value of the function $\Phi(L)$ not depend on the manner of ordering the edges of the graph L, it is necessary and sufficient that the conditions

$$(\alpha\mu - \mu\alpha)(\alpha + \mu)^m \mu^n (\alpha + \mu + 1) = 0; \tag{3_1}$$

$$(\alpha^2 - 1)\mu^{n+1}(\alpha + \mu + 1) = 0, \tag{3_2}$$

where $m, n = 0, 1, 2, \dots$, be fulfilled.

The method of proof of this assertion is borrowed from ([4]). Necessity is proved by comparing the values of Φ for the corresponding graph of Fig. 1 for two ways of ordering its edges. In deriving (3_2) we use the relation

Fig. 2

Figure 1: Fig. 2

$$(\alpha^m\mu^n - \mu^n\alpha^m)(\alpha+\mu+1) = 0, \tag{3_3}$$

which is an algebraic consequence of (3_1).

To prove sufficiency, we first show that, with the help of (3), the general expression $\Phi(L)$ is brought to the form

$$\Phi(L) = \{f_L(\alpha) + \alpha\varphi_L(\mu) + \psi_L(\mu)\}(\alpha+\mu+1) + \frac{1-(-1)^{d_2(L)}}{2}; \tag{4}$$

where

$$f_L(\alpha) = \begin{cases} \alpha^{d_2(L)-2} + \alpha^{d_2(L)-4} + \dots + \alpha^2 + 1, & \text{if } d_2(L) \equiv 0 \pmod 2, \\ \alpha^{d_2(L)-2} + \alpha^{d_2(L)-4} + \dots + \alpha^3 + \alpha, & \text{if } d_2(L) \equiv 1 \pmod 2, \end{cases} \tag{5}$$

and $\varphi_L(\mu)$, $\psi_L(\mu)$ are polynomials with integer nonnegative coefficients, moreover

$$\varphi_L(-\mu) = (-1)^{d_2(L)+1}\varphi_L(\mu); \qquad \psi_L(-\mu) = (-1)^{d_2(L)}\psi_L(\mu). \tag{6}$$

After this it remains to consider three cases of the mutual arrangement of the pair of edges *ac* and *bc* (or *bd*) in an arbitrary graph (Fig. 2); applying induction on the number of edges of the graph and using once again relations (3), we complete the proof of sufficiency.

Fig. 2

2. The expression (4) for the function $\Phi(L)$ is uniquely recovered from the general expression of the function $\Psi(L)$, satisfying the conditions

$$\Psi(L) = \Psi(L_\alpha) + \mu\Psi(L_\mu) + 1\,*; \tag{1'}$$

$$\Psi(E_n) = 0, \qquad n = 0, 1, 2, \dots \tag{2'}$$

Obviously, $\Psi(L)$ is a polynomial with integer nonnegative coefficients: $\Psi(L) = \sum_{k\geq 0} a_k(L)\mu^k$, moreover

$$a_k(L) = a_k(L_\alpha) + a_{k-1}(L_\mu), \qquad k = 1, 2, \dots; \tag{7}$$

$$a_0(L) = a_0(L_\alpha) + 1. \tag{8}$$

We shall now show that

$$a_k(L) = \sum_{i \geq 0} (-2)^i P_{i+k+1,i}(L), \tag{9}$$

where $P_{ji}(L)$ is the number of edge subgraphs M of the graph L for which $d_2(M) = j$ and $l(M) = i$.

It is easy to see that $P_{ji}(L) = P_{ji}(L_\alpha) + \overline{P}_{ji}(L)$, where $\overline{P}_{ji}(L)$ is the number of those edge subgraphs (with number of edges j and cyclomatic number i) of the graph L which contain the edge ab.

Let ρ be the number of triangles of the graph L containing the edge ab; the vertices of these triangles distinct from a and b will be called the c-vertices of the graph L.

Let $P_{ji}^{r,\varepsilon,s}(L_\mu)$ be the number of those edge subgraphs (with number of edges j and cyclomatic number i) of the graph L_μ whose composition includes r c-vertices of the graph L, of which ε belong to one connected component with the vertex $\overline{a}$ in the graph L_μ, while the remaining $r - \varepsilon$ are distributed among s connected components of the graph L_μ.

In any edge subgraph M of the graph L we distinguish classes of vertices belonging to the separate connected components of M itself or of the graph obtained from M as a result of the operation $L \to L_\mu$. Among the c-vertices of each of these classes we distinguish vertices adjacent to both a and b simultaneously, adjacent to only one of the vertices a, b, and, finally, not adjacent—

* It is clear that for $\alpha = 1$ the conditions (3) are fulfilled.

connected neither with a nor with b. The notation needed below for the numbers of c-vertices of one or another type is given in Fig. 3.

Fig. 3

Then

$$\bar{P}_{ji}(L) = \sum_{r=0}^{\rho} \sum_{\omega=0}^{r} \sum_{\omega_1=0}^{\omega} C_\omega^{\omega_1} 2^{\omega_1} \sum_{\varepsilon=0}^{r-\omega} \sum_{\varepsilon_1=0}^{\varepsilon} C_\varepsilon^{\varepsilon_1} 2^{\varepsilon_1} \sum_{\varepsilon_2=0}^{\varepsilon-\varepsilon_1} C_{\varepsilon-\varepsilon_1}^{\varepsilon_2} \times$$

$$\times \sum_{s \geq 0} \sum_{t=0}^{s} \left\{ \sum_{\sum_{n=1}^{t} q_n \leq r-\omega-\varepsilon} \prod_{n=1}^{t} \left[\sum_{q'_n=1}^{q_n} C_{q_n}^{q'_n} \sum_{q_{n1}=0}^{q'_n} C_{q'n}^{q_{n1}} 2^{q_{n1}} \right] \right\} P_{j_1,i_1}^{r-\omega,\varepsilon,s}(L_\mu),$$

where the expression in braces is regarded as equal to one for $t = 0$;

$$j_1 = j - 1 - (2\omega - \omega_1) - (\varepsilon_1 + 2\varepsilon_2) - \sum_{n=1}^{t}(2q'_n - q_{n1});$$

$$i_1 = i - (\omega - \omega_1) - (\varepsilon_1 + 2\varepsilon_2) - \sum_{n=1}^{t}(2q'_n - q_{n1}) + t.$$

Hence*

$$\sum_{i\geq 0}(-2)^i P_{i+k+1,i}(L) - \sum_{i\geq 0}(-2)^i P_{i+k+1,i}(L_\alpha) = \sum_{i\geq 0}(-2)^i \bar{P}_{i+k+1,i}(L) =$$

$$= \sum_{r=0}^{\rho}\sum_{\omega=0}^{r}\sum_{\omega_1=0}^{\omega} C_\omega^{\omega_1} 2^{\omega_1} \sum_{\varepsilon=0}^{r-\omega}\sum_{\varepsilon_1=0}^{\varepsilon} C_\varepsilon^{\varepsilon_1} 2^{\varepsilon_1} \sum_{\varepsilon_2=0}^{\varepsilon-\varepsilon_1} C_{\varepsilon-\varepsilon_1}^{\varepsilon_2} \times$$

$$\times \sum_{s\geq 0}\sum_{t=0}^{s}\left\{\sum_{\sum_{n=1}^t q_n \leq r-\omega-\varepsilon}\prod_{n=1}^{t}\left[\sum_{q'_n=1}^{q_n} C_{q_n}^{q'_n}\sum_{q_{n1}=0}^{q'_n} C_{q'n}^{q_{n1}} 2^{q_{n1}}\right]\right\} \times$$

$$\times \sum_{i_1\geq 0}(-2)^i P^{r-\omega,\varepsilon,s}_{i_1+k-\omega-t,\, i_1}(L_\mu) =$$

In replacing the summation index i_1 by i we used the fact that $P_{ji} = 0$ for $i < 0$.

$$= \sum_{r=0}^{\rho}\sum_{\omega=0}^{r}\left[\sum_{\omega_1=0}^{\omega} C_\omega^{\omega_1} 2^{\omega_1}(-2)^{\omega-\omega_1}\right]\sum_{\varepsilon=0}^{r-\omega}\left[\sum_{\varepsilon_1=0}^{\varepsilon} C_\varepsilon^{\varepsilon_1} 2^{\varepsilon_1}(-2)^{\varepsilon_1}\sum_{\varepsilon_2=0}^{\varepsilon-\varepsilon_1} C_{\varepsilon-\varepsilon_1}^{\varepsilon_2}(-2)^{2\varepsilon_2}\right] \times$$

$$\times \sum_{s\geq 0}\sum_{t=0}^{s}\left\{\sum_{\sum_{n=1}^t q_n \leq r-\omega-\varepsilon}\prod_{n=1}^{t}\left[\sum_{q'_n=1}^{q_n} C_{q_n}^{q'_n}(-2)^{q'_n}\left(\sum_{q_{n1}=0}^{q'_n} C_{q'n}^{q_{n1}} 2^{q_{n1}}(-2)^{q'_n-q_{n1}}\right)\right]\right\} \times$$

$$\times \sum_{i_1\geq 0}(-2)^{i_1-t} P^{r-\omega,\varepsilon,s}_{i_1+k-\omega-t,i_1}(L_\mu) = \sum_{r=0}^{\rho}\sum_{\varepsilon=0}^{r}\sum_{s\geq 0}\sum_{i_1\geq 0}(-2)^{i_1} P^{r,\varepsilon,s}_{i_1+k,i_1}(L_\mu) =$$

$$= \sum_{i_1\geq 0}(-2)^{i_1}\sum_{r,\varepsilon,s} P^{r,\varepsilon,s}_{i_1+k,i_1}(L_\mu) = \sum_{i_1\geq 0}(-2)^i P_{i_1+k,i_1}(L_\mu).$$

Thus

$$\sum_{i\geq 0}(-2)^i P_{i+k+1,i}(L) = \sum_{i\geq 0}{}' (-2)^i P_{i+k+1,i}(L_\alpha) + \sum_{i\geq 0}(-2)^i P_{i+k,i}(L_\mu), \qquad (7')$$

$$k = 1, 2, \ldots$$

Moreover,

$$P_{10}(L) = P_{10}(L_\alpha) + 1, \qquad (8')$$

for $P_{10}(L) = d_2(L)$. Comparing (7), (8) and (7′), (8′), and taking into account the uniqueness of the function $\Psi(L)$, we obtain (9).

Thus,

$$\Psi(L) = \sum_{i,k}(-2)^i P_{i+k+1,i}(L)\mu^k.$$

3. It is easy to show that the quantities $d_2(L)$ and $l(L)$ are connected by the relation

$$l(L) \leq d_2(L) + \tfrac{1}{2}\left[1 - \sqrt{d_2(L)+1}\right],$$

where the equality sign is attained if L is a graph almost isomorphic to a complete graph. It follows that edge subgraphs M of the graph L, for which $d_2(M) = j$ and $l(M) = i$, can exist only when

$$j \leq d_2(L); \qquad i \leq \min\{l(L);\ j + \tfrac{1}{2}(1 - \sqrt{8j+1})\}.$$

Taking this into account, we finally obtain

$$\psi(L) = \sum_{k=0}^{d_2(L)-l(L)-1} \sum_{i=0}^{{C_{k+1}}^2-1} (-2)^i P_{i+k+1,i}(L)\mu^k.$$

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Note: Figure translations are in progress. See original paper for figures.