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Abstract

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DETERMINATION OF PERIODIC MOTIONS OF CONSERVATIVE SYSTEMS WITH ONE DEGREE OF FREEDOM

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The article considers a new method for determining periodic motions of conservative systems, especially convenient for determining the period of oscillations and applicable in a broad domain.

Among nonlinear autonomous systems of the second order, the most thoroughly studied are conservative systems without friction, described by the equation

$$\ddot{x} + g(x) = 0. \tag{1}$$

If the function $g(x)$ is integrable, then the first integral of equation (1) is determined in an elementary way from the equality

$$\frac{1}{2}\dot{x}^2 = c - G(x), \tag{2}$$

where $G(x) = \int^x g(x)\,dx$; c is a constant of integration, equal to the kinetic energy of the system at the point at which $G(x) = 0$. However, determination of the second integral, especially the determination of periodic motions of the system described by equation (1), by integrating equation (2) is very difficult, except for certain special forms of the function $g(x)$. In most cases, when the function $g(x)$ is not nearly linear (2), and the second integral is not reducible to one of the elliptic integrals, one has to confine oneself to investigating the system by constructing phase trajectories on the phase plane $x, \dot{x}$ (1).

Investigation on the phase plane is a powerful tool for the qualitative study of the behavior of self-oscillatory systems; however, it is very laborious and, moreover, does not make it possible to determine the period of oscillation and to construct the oscillation curve of a nonlinear system in the plane x, t. It is therefore expedient to develop a method of integrating equation (2), applicable to a broad class of systems and allowing the period of oscillation of the system to be determined without difficulty. For the applicability of the method considered, we shall require that the function $G(x)$ be representable, with sufficient accuracy, by a polynomial in integral positive powers of x.

Let us make the change of variable $y = x - x_0$, where x_0 is a certain constant to be determined later. Then equation (1) takes the form

$$\ddot{y} + g(y + x_0) = 0. \tag{1a}$$

Now transform equation (2) to the form

$$\dot{y}^2 = n^2(A^2 - y^2) - n^2(A^2 - y^2) + c - 2G(y + x_0) \tag{2a}$$

or

$$\dot{y} = \pm n\sqrt{A^2 - y^2}\sqrt{1 - Q(y)}, \tag{2b}$$

where

$$Q(y) = 1 - \frac{[c - 2G(y + x_0)]}{n^2(A^2 - y^2)}; \tag{3}$$

n is an arbitrary real constant; A is a constant equal to the greatest deviation of the system from the point x_0 (from the point $y = 0$).

Examining equation (26) on the phase plane $y\dot{y}$, it is easy to verify that the phase trajectory on this plane forms a non-self-intersecting closed curve, corresponding to the periodicity of y, at least when the variable y varies in the closed interval $[-A \leq y \leq A]$, and the function $Q(y)$ satisfies in this interval the inequality

$$-\infty < Q(y) < 1. \tag{4}$$

It follows from (3) that, for inequality (4) to hold, it is necessary and sufficient that

$$0 < [c - 2G(y + x_0)]/n^2(A^2 - y^2) < \infty. \tag{5}$$

If inequality (5) is satisfied for some particular value of the constant n, then n can be chosen so that, in the closed interval $[-A, A]$, the inequality

$$0 < [c - 2G(y + x_0)]/n^2(A^2 - y^2) \leq 1 \tag{5a}$$

is also satisfied. Then, obviously,

$$0 \leq Q(y) < 1. \tag{4a}$$

Above we required that the function $G(x)$ be representable with sufficient accuracy in the form of a polynomial in powers of x; therefore, for any finite constant x_0, the function $G(y+x_0)$ will also be representable in the form of a polynomial in integral positive powers of y. Taking into account the arbitrariness of the constant x_0, we can choose it so that the function $G(y+x_0)$ satisfies the equality

$$2G(y+x_0) = B_0 + P(y) + (A^2 - y^2)\psi(y), \tag{6}$$

where $P(y)$ is a polynomial in even powers of y; $\psi(y)$ is a polynomial in odd powers of y; B_0 is a real constant or zero. In particular, if $g(x)$ is an odd function, then the constant x_0 is zero and $\psi(y) = 0$.

Next, taking into account the arbitrariness of the constant c, we can choose it so that the equality

$$c - P(y) = (A^2 - y^2)\Phi(y), \tag{7}$$

is satisfied, where $\Phi(y)$ is a polynomial in even powers of y, which may also contain a constant term. Then, in accordance with (3), $Q(y)$ is representable by the polynomial: $Q(y) = 1 - [\Phi(y) - \psi(y)]/n^2$, and inequality (5a) is transformed into the form

$$0 < \Phi(y) - \psi(y) \leq n^2. \tag{5b}$$

Proceeding to construct the solution of equation (26), we assume that inequality (5b), and consequently inequality (4a), are satisfied in the prescribed interval of variation of the variable y. Integrating (26), taking into account inequality (4a), we can determine with arbitrary accuracy the second integral of equation (1a) from the equality

$$t + \varphi_0 = -\frac{1}{n}\int_{-A}^{A}\left[1 + \frac{Q(y)}{2} + \frac{3}{8}Q^2(x) + \ldots\right]\frac{dy}{\sqrt{A^2 - y^2}}. \tag{8}$$

Suppose that, for the approximate integration of (26), k terms of series (8) have been taken (including unity), and denote the resulting partial sum by $S(k)$, i.e.

$$\frac{1}{\sqrt{1 - Q(y)}} - S(k) = \sum_{i=k}^{\infty}\gamma_i Q^i(y), \tag{9}$$

where γ_i are the binomial coefficients of the expansion (all positive). Further, let Q_m and S_m be the maximum values of $Q(y)$ and $S(k)$, respectively—

but for oscillations of the system with a prescribed amplitude A (relative to the point $y = 0$). Then, taking (4a) and (9) into account, by elementary operations we obtain

$$[1/\sqrt{1-Q_m} - S_m]\, Q^k(y)/Q_m^k \geq 1/\sqrt{1-Q(y)} - S(k) \geq 0. \tag{10}$$

From this it is easy to estimate the error of the approximate integration of the equation by means of equality (8). With the aid of (8) one can graphically construct the oscillation curve over a period in the coordinates x, t.

In the investigation of periodic motions of conservative systems without friction in applied mechanics, an often important problem is the determination of the period of oscillations. Approximately, the period of oscillations of the system is determined from the equation

$$T_0 = -\frac{2}{n} \int_{-A}^{A} \frac{S(k)}{\sqrt{A^2 - y^2}}\, dy. \tag{11}$$

Since $Q^k(y)/\sqrt{A^2 - y^2}$ is an integrable function on the interval $(-A, A)$, taking (10) into account, for estimating the exact period of oscillations we obtain the inequality

$$T \leq -\frac{2}{n} \int_{-A}^{A} \left\{ S(k) + \left[\frac{1}{\sqrt{1-Q_m}} - S_m \right] \frac{Q^k(y)}{Q_m^k} \right\} \frac{dy}{\sqrt{A^2 - y^2}}. \tag{12}$$

Therefore the error in determining the period from equation (11) is estimated by the inequality

$$0 \leq T - T_0 \leq -\frac{2}{n} \int_{-A}^{A} \left[\frac{1}{\sqrt{1-Q_m}} - S_m \right] \frac{Q^k(y)}{Q_m^k} \frac{dy}{\sqrt{A^2 - y^2}}. \tag{13}$$

The partial sum $S(k)$ can be represented in the form

$$S(k) = a_0 + \sum_{i=1}^{n} (a_i y^{2i} + b_i y^{2i-1}). \tag{14}$$

Taking into account that

$$\int_{A}^{-A} \sum_{i=1}^{n} (a_i y^{2i} + b_i y^{2i-1}) \frac{dy}{\sqrt{A^2 - y_i^2}} = \pi \sum_{i=1}^{n} a_i A^{2i} \frac{1 \cdot 3 \cdot 5 \ldots (2i-1)!}{2 \cdot 4 \cdot 6 \ldots 2i!}, \tag{15}$$

from (11) and (14) we obtain:

$$T_0 = \frac{2\pi}{n}\left[a_0 + \sum_{i=1}^{n} a_i A^{2i} \frac{1\cdot 3\cdot 5\ldots(2i-1)!}{2\cdot 4\cdot 6\ldots 2i!}\right]. \tag{16}$$

As an example, let us consider the equation:

$$\ddot{x} + \alpha_0 + \alpha_1 x + \alpha_3 x^3 = 0. \tag{17}$$

Putting $x = y + x_0$, we find the first integral; after elementary transformations we obtain

$$\dot{y}^{\,2} = n^2(A^2{-}y^2)\left[1 - 1 + \frac{c - 2(\alpha_0 + \alpha_1 x_0 + x_0^3)y - (\alpha_1 + 3\alpha_3 x_0)y^2 - 2\alpha_3 x_0 y^3 - 0.5\alpha_3 y^4}{n^2(A^2 - y^2)}\right].$$

Further, determining the constant x_0 from the equality

$$x_0^3 + (\alpha_1 + {}^2/_2 A^2\alpha_3)x_0 + \alpha_0 = 0 \tag{18}$$

and the constant c from the equality $c = \alpha_1 A^2 + 0.5\alpha_3 A^4$, we obtain

$$\dot{y} = \pm n\sqrt{A^2 - y^2}\sqrt{1 - 1 + \frac{1}{n^2}\left[\alpha_1 + 0.5\alpha_3 x_0 y + 0.5\alpha_3(A^2 + y^2)\right]}, \tag{19}$$

On the basis of what has been set forth, one may assert that the general solution of equation (17) is periodic at least when the inequality

$$0.5a_3\left[A^2 + y^2 + x_0 y\right] + a_1 > 0. \tag{20}$$

is satisfied.

In particular, if the constants a_3 and a_1 are positive, and the constant x_0, determined from equality (18), satisfies the inequality $x_0^2 \leqslant 4A^2$, then inequality (20) will be satisfied for arbitrary amplitudes A. If the constant a_1 is positive and a_3 is negative, then inequality (20) will be satisfied only if the inequality $A^2 < |2a_1/a_3(2A^2{+}x_0A)|$ is satisfied. Finally, if a_1 is negative and a_3 is positive, then inequality (20) will be satisfied only if the inequality $A^2 > |2a_1/a_3|$ is satisfied for positive x_0, etc.

As is seen from this example, establishing a criterion for the existence of periodic motions of the system described by equation (1) is very simple.

Next put $a_0 = a_1 = 0$, $a_3 > 0$, and $n^2 = a_3 A^2$. Then, in accordance with (18), for the constant x_0 we have the trivial solution $x_0 = 0$ and $y = X$. From (19) we obtain

$$\dot{x} = \pm\sqrt{a_3}A\sqrt{A^2 - x^2}\sqrt{1 - (A^2 - x^2)/2A^2}.$$

Putting $k = 3$ in $S(k)$, in accordance with (19a) we obtain

$$T_0 = \frac{1}{16A\sqrt{a_3}} \int\limits_{A}^{-A} \left[43 - \frac{14x^2}{A^2} + \frac{3x^4}{A^4}\right] \frac{dx}{\sqrt{A^2 - x^2}} = \frac{2.320\pi}{A\sqrt{a_3}}. \qquad (21)$$

We estimate the error in determining the period from (21), in accordance with (13), from the inequality

$$0 < T - T_0 < -\frac{2}{A\sqrt{a_3}} \int\limits_{-A}^{A} \left(\sqrt{2} - \frac{43}{32}\right) \frac{(A^2 - x^2)^3}{A^6} \frac{dx}{\sqrt{A^2 - x^2}}$$

or

$$0 < T - T_0 < 0.05\,\pi/A\sqrt{a_3}.$$

Then put

$$T_1 = (2.320 + 0.025)\pi/A\sqrt{a_3},$$

and then $|T - T_1| < 0.025\pi/A\sqrt{a_3}$, and the error in determining the period will be no more than 1%. If one puts $k = 4$ in $S(k)$, then the error in determining the period will be no more than 0.4%, etc.

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CITED LITERATURE

1. A. A. Andronov, A. A. Vitt, S. E. Khaikin, *Theory of Oscillations*, 1959.
2. N. N. Bogolyubov, Yu. A. Mitropolsky, *Asymptotic Methods in the Theory of Nonlinear Oscillations*, 1958.

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