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Abstract

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ON A RIESZ BASIS OF EXPONENTIAL FUNCTIONS

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A basis $\{x_k\}$ of a Hilbert space H will be called, following N. K. Bari (^{1}), a **Riesz basis** if the series

$$\sum_{k=-\infty}^{\infty} c_k x_k,$$

where c_k $(k = 0, \pm 1, \ldots)$ are numbers, converges if and only if $\{c_k\} \in l^2$.

As follows directly from the Riesz-Fischer theorem and Parseval' s equality, the trigonometric system $\{e^{ikt}\}$ is, by virtue of the periodicity of its elements, a Riesz basis in its closed linear span relative to the space $L^2(-\sigma, \sigma)$ for any $\sigma \geq \pi$.

The purpose of the present work is to study conditions under which an analogous fact holds for an arbitrary system of exponential functions.

1. A system $\{x_k\}$ of elements of a Hilbert space H is usually called **uniformly minimal** (^{1,2}) if in H there exists a norm-bounded system $\{h_k\}$ conjugate to $\{x_k\}$, i.e. such that $(x_k, h_j) = \delta_{kj}$. A Riesz basis is, obviously, a uniformly minimal system.

Let Λ be an arbitrary sequence of complex numbers λ_k $(k = 0, \pm 1, \ldots)$, and let E_Λ be the corresponding system of exponential functions $\{e^{i\lambda_k t}\}$. By $\tau = \tau(\Lambda)$ we shall denote the exact lower bound of the numbers $\sigma > 0$ such that the system E_Λ is uniformly minimal in $L^2(-\sigma, \sigma)$.

If the system E_Λ is uniformly minimal in $L^2(-\sigma, \sigma)$ for every $\sigma > 0$, then $\tau = 0$; if, however, it is not uniformly minimal in $L^2(-\sigma, \sigma)$ for any $\sigma > 0$, then naturally one sets $\tau = \infty$.

The system E_Λ may be in $L^2(-\tau, \tau)$ both uniformly minimal and nonminimal even in the ordinary sense. Indeed, the trigonometric system $\{e^{ikt}\}$ is uniformly minimal in $L^2(-\pi, \pi)$ and is not minimal in $L^2(-\sigma, \sigma)$ for any $\sigma < \pi$. On the other hand, the system E_Λ with $\lambda_k = k - \frac{1}{4}\operatorname{sgn} k$ $(k = 0, \pm 1, \ldots)$ is uniformly minimal in $L^2(-\sigma, \sigma)$ for every $\sigma > \pi$ (this follows from Theorem 2), but is not minimal even in the ordinary sense in $L^2(-\pi, \pi)$ (^{3}).

2. **Definition.** A sequence Λ of complex numbers λ_k $(k = 0,\ \pm 1, ...)$ **belongs to the class** K if

$$\sup_{-\infty<k<\infty} |\operatorname{Im}\lambda_k| < \infty; \tag{1}$$

$$\inf_{-\infty<m,n<\infty} |\lambda_m - \lambda_n| > 0 \quad (m \neq n). \tag{2}$$

We shall also denote by $L^2_\Lambda(-\sigma,\sigma)$ the closed linear span of the system E_Λ in $L^2(-\sigma,\sigma)$.

Theorem 1. *If the system E_Λ is a Riesz basis in $L^2_\Lambda(-\sigma,\sigma)$ for some $\sigma > 0$, then the sequence Λ belongs to the class K and $\sigma \geq \tau$.*

Proof. Since the series

$$\sum_{-\infty}^{\infty} c_k \int_{-\sigma}^{\sigma} f(t)e^{i\lambda_k t}\, dt$$

converges for every function $f(t) \in L^2(-\sigma,\sigma)$ and every sequence $\{c_k\} \in l^2$, it follows that

$$\sum_{-\infty}^{\infty} \left| \int_{-\sigma}^{\sigma} f(t)e^{i\lambda_k t}\, dt \right|^2 < \infty$$

and, consequently,

$$\frac{\operatorname{sh}(2\sigma \operatorname{Im}\lambda_k)}{\operatorname{Im}\lambda_k} = \int_{-\sigma}^{\sigma} \left|e^{i\lambda_k t}\right|^2\, dt = O(1),$$

whence, as $k \to \infty$, $\operatorname{Im}\lambda_k = O(1)$.

Further, the system E_Λ is uniformly minimal in $L^2(-\sigma,\sigma)$, and therefore it has an adjoint system $\{h_k(t)\}$, bounded in norm. The equality

$$\int_{-\sigma}^{\sigma} \left(e^{i\lambda_k t} - e^{i\lambda_j t}\right) \overline{h_k(t)}\, dt = 1 \qquad (k \neq j)$$

shows that the limiting relation

$$\lim_{k,j\to\infty} |\lambda_k - \lambda_j| = 0 \qquad (k \neq j)$$

is impossible. Thus, the theorem is completely proved.

Theorem 2. If the sequences $\Lambda = \{\lambda_k\}$ and $M = \{\mu_k\}$ both belong to the class K and $\lambda_k = \mu_k + O(1)$ as $k \to \infty$, then $\tau(\Lambda) = \tau(M)$.

We do not give the proof of this theorem here for lack of space.

Corollary. If the sequence Λ belongs to the class K, then $\tau(\Lambda) < \infty$.

Indeed, every sequence Λ of class K is contained in some sequence of the form $\{\alpha k + O(1)\}$, where α is a constant, and for such a sequence, by Theorem 2, $\tau < \infty$.

3. Theorem 3. If the sequence Λ belongs to the class K, then the system E_Λ is a Riesz basis in $L^2_\Lambda(-\sigma, \sigma)$ for every $\sigma > \tau(\Lambda)$.

For $\sigma = \tau$ the assertion of the theorem, generally speaking, does not hold, since in $L^2(-\tau, \tau)$ the system E_Λ may fail to be even minimal.

We precede the proof of Theorem 3 by several auxiliary propositions.

Lemma 1. If the sequence Λ belongs to the class K, then for every $\sigma > 0$ there exists a constant B such that

$$\int_{-\sigma}^{\sigma} \left|\sum c_k e^{i\lambda_k t}\right|^2 dt \leq B \sum |c_k|^2$$

for every finite system of complex numbers c_k.

Proof. Let n_k be the integer nearest to λ_k. We may assume from the outset that $n_k \neq n_j$ for $k \neq j$, since the whole sequence Λ can be split into a finite number of subsequences possessing this property.

First of all it is clear that

$$\int_{-\sigma}^{\sigma} \left|\sum c_k e^{i\lambda_k t}\right|^2 dt = \sum_{k \neq j} \sum c_k \overline{c_j} \int_{-\sigma}^{\sigma} e^{i(\lambda_k - \overline{\lambda_j})t}\, dt + \sum_k |c_k|^2 \int_{-\sigma}^{\sigma} \left|e^{i\lambda_k t}\right|^2 dt,$$

where the last integral on the right-hand side has a finite upper bound with respect to k. Integrating by parts, we obtain

$$\int_{-\sigma}^{\sigma} e^{i(\lambda_k - \overline{\lambda_j})t}\, dt = \frac{1}{n_k - n_j} \left\{ -ie^{i(\lambda_k - \overline{\lambda_j})t} \Big|_{-\sigma}^{\sigma} - \int_{-\sigma}^{\sigma} (\delta_k - \overline{\delta_j}) e^{i(\lambda_k - \overline{\lambda_j})t}\, dt \right\},$$

where $\delta_k = \lambda_k - n_k = O(1)$ as $k \to \infty$. To complete the proof, it remains to···

it remains now only to apply Hilbert’ s theorem, by virtue of which

$$\left|\sum_{k\neq j}\sum \frac{A_{kB}j}{k-j}\right| < 2\pi\left\{\sum |A_k|^2 \sum |B_j|^2\right\}^{1/2}$$

for arbitrary complex A_k, B_j (^{5}).

Lemma 2. *If the sequence Λ belongs to the class K, then for any sequence $\{c_k\} \in l^2$ there exists an entire function $F(\lambda)$ of exponential type not exceeding a given number $\sigma > \tau(\Lambda)$, such that $F(\lambda_k) = c_k$ $(k = 0, \pm 1, ...)$ and*

$$\int_{-\sigma}^{\sigma} |F(x)|^2\, dx \le M \sum_{-\infty}^{\infty} |c_k|^2,$$

where the constant M does not depend on the sequence $\{c_k\}$.

Proof. Put, for $\tau < \sigma_0 < \sigma,\ 0 < \omega < \sigma - \sigma_0$,

$$F(\lambda) = \sum_{-\infty}^{\infty} c_k \frac{\sin \omega(\lambda - \lambda_k)}{\omega(\lambda - \lambda_k)} \int_{-\sigma_0}^{\sigma_0} \overline{h_k(t)} e^{i\lambda t}\, dt,$$

where $\{h_k(t)\}$ is the system from $L^2_\Lambda(-\sigma_0, \sigma_0)$ biorthogonal to E_Λ. It is clear that $F(\lambda)$ is an entire function of exponential type not greater than σ, and $F(\lambda_k) = c_k$ $(k = 0, \pm 1, ...)$. By Cauchy' s inequality,

$$|F(\lambda)|^2 \le \sum_{n=-\infty}^{\infty} \left|\frac{\sin \omega(\lambda - \lambda_n)}{\omega(\lambda - \lambda_n)}\right|^2 \sum_{k=-\infty}^{\infty} \left| c_k \int_{-\sigma_0}^{\sigma_0} \overline{h_k(t)} e^{i\lambda t} dt \right|^2.$$

Since the first sum on the right-hand side has a finite upper bound for all real λ, it follows that

$$\int_{-\infty}^{\infty} |F(x)|^2\, dx \le M_1 \sum_{-\infty}^{\infty} |c_k|^2 \int_{-\sigma_0}^{\sigma_0} |h_k(t)|^2\, dt \le M \sum_{-\infty}^{\infty} |c_k|^2,$$

where M, obviously, does not depend on the sequence $\{c_k\}$.

Lemma 3. *If the sequence Λ belongs to the class K, then for every $\sigma > \tau(\Lambda)$ there exists a constant A such that*

$$A \sum |c_k|^2 \le \int_{-\sigma}^{\sigma} \left|\sum c_{ke}^{i\lambda_k t}\right|^2\, dt$$

for every finite system of complex numbers c_k.

Proof. Let the numbers c_k be included in some sequence $\{c_k\} \in l^2$. By Lemma 2 there exists a function $f(t) \in L^2(-\sigma, \sigma)$ such that

$$\int_{-\sigma}^{\sigma} f(t) e^{i\lambda_k t}\, dt = \overline{c_k} \qquad (k = 0, \pm 1, \ldots),$$

therefore,

$$\sum_{-\infty}^{\infty} |c_k|^2 = \int_{-\sigma}^{\sigma} f(t) \sum_{-\infty}^{\infty} c_{ke}^{i\lambda_k t}\, dt,$$

and, on the basis of Bunyakovsky’ s inequality, the right-hand side does not exceed the magnitude of the expression

$$\left\{ \int_{-\sigma}^{\sigma} |f(t)|^2\, dt \int_{-\sigma}^{\sigma} \left| \sum_{-\infty}^{\infty} c_{ke}^{i\lambda_k t} \right|^2 dt \right\}^{1/2}$$

and, moreover,

$$\int_{-\sigma}^{\sigma} |f(t)|^2\, dt \leq M \sum_{-\infty}^{\infty} |c_k|^2.$$

Thus the lemma is completely proved.

4. **Proof of Theorem 3***. Let $f(t) \in L^2_\Lambda(-\sigma, \sigma)$, i.e.

$$\lim_{n\to\infty} \int_{-\sigma}^{\sigma} \left| f(t) - \sum_{k=-n}^{n} c_{kn} e^{i\lambda_k t} \right|^2 dt = 0$$

for some complex numbers c_{kn} $(k = 0,\ \pm 1, \ldots, \pm n)$. Put

$$P_n(t) = \sum_{k=-n}^{n} c_{kn} e^{i\lambda_k t}; \qquad S_n(t) = \sum_{k=-n}^{n} e^{i\lambda_k t} \int_{-\sigma}^{\sigma} f(t) \overline{h_k(t)}\, dt,$$

where $\{h_k(t)\}$ is the system in $L^2_\Lambda(-\sigma, \sigma)$ conjugate to E_Λ. Then

$$\int_{-\sigma}^{\sigma} |f(t) - S_n(t)|^2\, dt \leq 2 \int_{-\sigma}^{\sigma} |f(t) - P_n(t)|^2\, dt + 2 \int_{-\sigma}^{\sigma} |S_n(t) - P_n(t)|^2\, dt,$$

and the second integral on the right-hand side is equal to the limit, as $N \to \infty$, of the expression

$$\int_{-\sigma}^{\sigma}\left|\sum_{k=-n}^{n}(c_{kN}-c_{kn})e^{i\lambda_k t}\right|^2 dt \leq B\sum_{k=-N}^{N}|c_{kN}-c_{kn}|^2.$$

The latter sum, in turn, does not exceed, up to a constant factor, the value of the integral

$$\int_{-\sigma}^{\sigma}\left|\sum_{k=-N}^{N}(c_{kN}-c_{kn})e^{i\lambda_k t}\right|^2 dt = \int_{-\sigma}^{\sigma}|P_N(t)-P_n(t)|^2\,dt.$$

Thus,

$$\int_{-\sigma}^{\sigma}|f(t)-S_n(t)|^2\,dt \leq M\int_{-\sigma}^{\sigma}|f(t)-P_n(t)|^2\,dt,$$

i.e. E_Λ is a basis in $L^2_\Lambda(-\sigma,\sigma)$. That this basis is a Riesz basis follows from Lemmas 1 and 3.

We shall indicate some consequences of the theorem proved.

Corollary 1. If $|\lambda_n - n| \leq D < \infty$ $(n = 0,\ \pm 1, \ldots)$ and the sequence Λ belongs to the class K, then E_Λ is a Riesz basis in $L^2_\Lambda(-\sigma,\sigma)$ for every $\sigma > \pi$.**

Corollary 2. If the sequence Λ belongs to the class K, then for every function $f(t) \in L^2_\Lambda(-\sigma_0,\sigma_0)$, $\sigma_0 > \tau$, there exists, for every $\sigma > \sigma_0$, a unique function $g(t) \in L^2_\Lambda(-\sigma,\sigma)$ such that $g(t) = f(t)$ almost everywhere on the interval $(-\sigma_0,\sigma_0)$.***

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* On the basis of results of N. K. Bari (1), the assertion of Theorem 3 follows directly from Lemmas 1 and 3. We give the proof only because this fact is elementary.

** In a number of works, conditions were studied under which the system E_Λ, where $|\lambda_k - k| < D$, is a Riesz basis in $L^2(-\pi, \pi)$ (see (6-8)).

*** For similar facts see (6, 9, 10).

Note: Figure translations are in progress. See original paper for figures.

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