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Abstract

Full Text

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UNIQUENESS OF THE SOLUTION OF THE INVERSE PROBLEM FOR ORDINARY DIFFERENTIAL OPERATORS OF ORDER $n \geq 2$ AND TRANSFORMATIONS OF SUCH OPERATORS

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Let L be a linear differential operator of the form

$$Ly = y^{(n)} + \sum_{k=0}^{n-2} p_k(x) y^{(k)}, \tag{1}$$

where $p_k^{(k)}(x)$ $(0 \leq x \leq 1)$ are bounded; denote by Λ the class of operators of the form (1). For $n = 2$ the inverse problem of spectral analysis, i.e. the problem of determining an operator from its spectral data, was solved by I. M. Gel' fand, M. G. Krein, B. M. Levitan, and V. A. Marchenko ([1–3]). In these investigations an important role is played by Volterra transformations of operators (1). In the case of entire analytic coefficients $p_k(x)$, Volterra transformations also exist for $n > 2$ and were applied by L. A. Sakhnovich ([4]); for nonanalytic $p_k(x)$, when $n > 2$, Volterra transformations, generally speaking, do not exist, as was established by V. I. Matsayev ([5]). Here, for operators from Λ with $n \geq 2$, another method is proposed which made it possible to prove uniqueness of the solution of the inverse problem, and transformations of operators (1), different from those constructed by M. K. Fage ([6]), are built.

The question of transformation plays the fundamental role in the problem of reconstructing an operator from spectral data.

1°. First we describe the spectral boundary-value problems considered for the operator L and prove uniqueness of the solution of the inverse problem. Preliminarily, for an arbitrary function $F(x)$ denote by $\vec{F}(x)$ the vector in the n-dimensional complex vector space E^n with components $F(x), F^1(x), \dots, F^{(n-1)}(x)$. We introduce boundary conditions by means of linear forms C_ν^a in E^n, where $\nu = 1, \dots, n$ and $a = 0$ or 1, and for the vector $z = (z_0, \dots, z_{n-1}) \in E^n$

$$C_\nu^a z = z_{\nu-1} + \sum_{0\le\mu\le\nu-2} c_{\nu\mu}^a z_\mu \qquad (c_{\nu\mu}^a = \text{const};\ 1\le\nu\le n;\ a=0,1). \quad (2)$$

Formulation of the system of spectral boundary-value problems. For $1 \le k \le n-1$ we consider the spectral boundary-value problem

$$L\varphi = \lambda\varphi, \qquad C_\nu^0\vec{\varphi}(0) = 0 \quad (k<\nu\le n); \qquad C_{\nu'}^1\vec{\varphi}(1) = 0 \quad (n-k<\nu'\le n). \quad (3)$$

The spectral boundary-value problem adjoint to (3) for the adjoint operator $\widetilde{L}$ is specified by means of the linear forms $\widetilde{C}_\nu^a$ in E^n. By λ_{kl}, $\varphi_{kl}(x)$, and $f_{kl}(x)$ $(l=1,2,...)$ we denote, respectively, the eigenvalues and the unnormalized eigenfunctions of problem (3) and of the adjoint problem,

$$|\lambda_{k1}| \le |\lambda_{k2}| \le \cdots,$$

$$\alpha_{kl} = \frac{1}{C_k^0\vec{\varphi}_{kl}(0)\,\widetilde{C}_k^0 f_{kl}(0)} \int_0^1 \varphi_{kl}(t)\,\overline{f}_{kl}(t)\,dt. \quad (4)$$

Definition 1. The totality of numbers $\{\lambda_{kl},\alpha_{kl};\ 1\le k\le n-1;\ l=1,2,...\}$ is called the $\{C_\nu^a\}$-spectral characteristic of the operator L. The functions $\varphi_{kl}(x)$, $f_{kl}(x)$ $(1\le k\le n-1;\ l=1,2,...)$ are called the $\{C_\nu^a\}$-eigenfunctions of the operators L and $\widetilde{L}$. Here $\{C_\nu^a\}$ is the system consisting of the forms $C_1^0,\dots,C_n^0,C_1^1,\dots,C_n^1$.

At the same time, for simplicity we assume* that:

α) All eigenvalues λ_{kl} are simple and are distinct from one another for different k.

For $1\le k\le n-1$, the complex numbers λ_{kl},α_{kl} $(l=1,2,...)$ determine the spectral measure $d\sigma_k(\lambda)$ by the formula

$$\int F(\lambda)\,d\sigma_k(\lambda) = \sum_l \frac{1}{\alpha_{kl}} F(\lambda_{kl})$$

and the corresponding Parseval equality is satisfied.

Theorem 1. *The $\{C_\nu^a\}$-spectral characteristic of an operator L uniquely determines the operator $L\in\Lambda$ and the forms C_ν^a of the form (2).*

We outline the proof. Suppose that $\{C_\nu^a\}$ is the spectral characteristic of the operator L, equal to $\{D_\nu^a\}$, the spectral characteristic of the operator L_1. For complex λ, define the functions $\varphi_{r\lambda}(x)$ and $\psi_{r\lambda}(x)$ $(r=1,\dots,n)$ by the relations

$$L\varphi_{r\lambda} = \lambda\varphi_{r\lambda}, \qquad C_\nu^0\vec{\varphi}_{r\lambda}(0) = 0, \qquad C_r^0\vec{\varphi}_{r\lambda}(0) = 1, \qquad C_\nu^1\varphi_{r\lambda}(1) = 0; \quad (5)$$

$$L_1\psi_{r\lambda} = \lambda\psi_{r\lambda}, \qquad D^0_\nu\vec{\psi}_{r\lambda}(0) = 0, \qquad D^0_r\vec{\psi}_{r\lambda}(0) = 1, \qquad D^1_\nu\psi_{r\lambda}(1) = 0. \quad (6)$$
$$(r < \nu \leq n, \quad n - r + 1 < \nu' \leq n).$$

Define the (n, n)-matrix $T^c_\lambda(x)$ by the equality

$$T^c_\lambda(x)\vec{\varphi}_{r\lambda}(x) = \vec{\psi}_{r\lambda}(x) \qquad (1 \leq r \leq n). \quad (7)$$

Computing the poles of the functions $\varphi_{r\lambda}(x)$ and $\psi_{r\lambda}(x)$ as analytic functions of λ, we obtain that $T^c_\lambda(x)$ is an entire analytic function of λ. From the asymptotics of $\varphi_{r\lambda}(x)$ and $\psi_{r\lambda}(x)$ it follows that $T^c_\lambda(x) = o(|\lambda|)$ as $\lambda \to \infty$. Therefore $T^c_\lambda(x)$ does not depend on λ; hence from (5)–(7) it follows readily that $L = L_1$ and $C^a_\nu = D^a_\nu$ for all ν and a.

Definition 2. Let L be a self-adjoint operator and, for $\alpha = 1, \ldots, n-1$, let problem (3) with $k = \alpha$ be adjoint to problem (3) with $k = n - \alpha$. Then the system of forms $\{C^a_\nu\}$ is called self-adjoint.

Theorem 2. *An operator L and a system of forms $\{C^a_\nu\}$ of the form (2) are self-adjoint if and only if*

$$\lambda_{n-k,l} = \overline{\lambda}_{k,l}, \qquad \alpha_{n-k,l} = \overline{\alpha}_{k,l} \qquad (1 \leq k \leq n-1;\ l = 1, 2, \ldots),$$

where $\{\lambda_{kl}, \alpha_{kl}\}$ is the $\{C^a_\nu\}$-spectral characteristic of the operator L.

For $n = 2$, Theorems 1 and 2 yield the known theorem that the spectral function of the Sturm–Liouville equation on a finite interval uniquely determines the equation and the boundary conditions.

2°. We now construct transformations of the operator L into other operators from Λ. We construct these transformations with the aid of a system of numbers δ^A_i and functions $\varphi_i(x)$ and $f_i(x)$ satisfying the relations

$$\sum_i |\delta^A_i - 1| < \infty, \qquad L\varphi_i = \lambda_i\varphi_i, \qquad \widetilde{L}f_i = \overline{\lambda}_i f_i, \qquad [\varphi_i, \tilde{f}_i] = 0, \quad (8)$$

where the index i runs through a countable set of values, λ_i are complex numbers, and $[\varphi, \tilde{f}](x)$ is the Lagrange form of the operator L. We shall also assume that the functions $\varphi_i(x)$ and $\tilde{f}_i(x)$ satisfy the following condition:

β) There exist nonnegative bounded functions $\tilde{\varphi}_i(x)$ and $\tilde{f}_i(x)$ such that the series

$$\sum_i \left(1 + |\lambda_i|^{\frac{n-1}{n}}\right) \tilde{\varphi}_i(x)\tilde{f}_i(x)$$

converges uniformly, and

* This restriction and restriction (2), imposed on the forms C_ν^a, are inessential.

the inequalities hold:

$$|\varphi_i^{(k)}(x)| \leq (1+|\lambda_i|^{k/n})\widetilde{\varphi}_i(x), \qquad |f_i^{(k)}(x)| \leq (1+|\lambda_i|^{k/n})\tilde{f}_i(x) \quad (k=0,\dots,n);$$

$$\left|\int_0^x \varphi_i(t)\overline{f_i(t)}\,dt\right| \leq \widetilde{\varphi}_i(x)\tilde{f}_i(x),$$

$$|[\varphi_i, f_j](x)| \leq |\lambda_i-\lambda_j|\widetilde{\varphi}_i(x)\tilde{f}_j(x) \quad (i\neq j,\ 0\leq x\leq 1).$$

Preliminarily, with the aid of φ_i, f_i, and δ_i^A, we form the infinite matrix

$$A(x) = \|A_{ij}(x)\|,$$

$$A_{ij}(x) = \frac{[\varphi_i, f_j](x)}{\lambda_i-\lambda_j} \quad \text{for } i\neq j, \qquad A_{ii}(x) = \delta_i^A + \int_0^x \varphi_i(t)\overline{f_i(t)}\,dt. \tag{9}$$

We shall call $A(x)$ the $\{\delta_i^A, \varphi_i, f_i\}$-matrix; $\{\delta_i^A, \varphi_i, f_i\}$ denotes the totality of the numbers δ_i^A and the functions $\varphi_i(x)$ and $f_i(x)$, corresponding to all possible i. Using (7), one can prove that the infinite determinant $\det A(x)$ converges and depends continuously on x. Assuming that

$$\det A(x) \neq 0 \qquad (0\leq x\leq 1), \tag{10}$$

we construct the matrix

$$[A(x)]^{-1} = B(x) = \|B_{ij}(x)\| \tag{11}$$

and the functions

$$\psi_i(x) = \sum_j B_{ij}(x)\varphi_j(x); \qquad g_i(x) = -\sum_j \overline{B_{ji}(x)}f_j(x). \tag{12}$$

Lemma 1. If conditions (8), β), and (10) are satisfied, then there exists an operator $L_1 \in \Lambda$ such that, for all i,

$$L_1\psi_i = \lambda_i\psi_i, \qquad \widetilde{L}_1 g_i = \overline{\lambda_i}g_i, \qquad [\psi_i, g_i] = 0,$$

where $\psi_i(x)$ and $g_i(x)$ are determined by formulas (12), (11), (9); $\widetilde{L}_1$ is the operator adjoint to L_1; $[\psi, g]_1(x)$ is the Lagrange form of the operator L_1.

Definition 3. The $\{\delta_i^A, \varphi_i, f_i\}$-transformation of the operator L is the operator L_1 indicated in Lemma 1.

Theorem 3. Under the conditions of Lemma 1, if

$$\delta_i^B = \overline{B_{ii}(0)},$$

then $B(x)$ is the $\{\delta_i^B, \psi_i, g_i\}$-matrix, and L is the $\{\delta_i^B, \psi_i, g_i\}$-transformation of the operator L_1. There exists a triangular (n,n)-matrix $T(x)$ such that, if $\lambda \neq \lambda_i$ for all i, then the formula

$$\vec{\psi}_\lambda(x) = T(x)\vec{\varphi}_\lambda(x) - \sum_i \frac{[\varphi_\lambda, f_i](x)}{\lambda - \lambda_i}\vec{\psi}_i(x) \tag{13}$$

establishes a one-to-one correspondence between the functions $\varphi_\lambda(x)$ such that $L\varphi_\lambda = \lambda\varphi_\lambda$, and the functions $\psi_\lambda(x)$ such that $L_1\psi_\lambda = \lambda\psi_\lambda$.

Formula (13) can be written with the aid of (n,n)-matrices $T_\lambda(x)$ and $Q_i(x)$:

$$\vec{\psi}_\lambda(x) = T_\lambda(x)\vec{\varphi}_\lambda(x), \qquad T_\lambda(x) = T(x) - \sum_i \frac{Q_i(x)}{\lambda - \lambda_i}; \tag{14}$$

$$Q_i(x)\vec{\varphi}(x) = [\varphi, f_i](x)\vec{\psi}_i(x) \tag{15}$$

for an arbitrary $\varphi(x)$.

3°. **Definition 4.** The $\{\varphi_{kl}, f_{kl}\}$-transformation is the $\{\delta_i^A, \varphi_i, f_i\}$-transformation of the operator L such that the index $i = (k,l)$,

$\lambda_i = \lambda_{kl}$, $\psi_i = \varphi_{kl}$, $f_i = f_{kl}$, and $\delta_i^A = 1$, where $k = 1, \dots, n-1$; $l = 1, 2, \dots$; $\varphi_{kl}(x)$ and $f_{kl}(x)$ are the $\{C_\nu^a\}$-eigenfunctions of the operators L and $\widetilde{L}$, normalized arbitrarily; λ_{kl} are the corresponding eigenvalues; $\{\varphi_{kl}, f_{kl}\}$ denotes the totality of the functions $\varphi_{kl}(x)$ and $f_{kl}(x)$, corresponding to all possible k and l.

In the case of $\{\varphi_{kl}, f_{kl}\}$-transformations, the matrices $A(x)$, $T_\lambda(x)$, and $T(x)$, introduced in §2 for arbitrary $\{\delta_i^A, \varphi_i, f_i\}$-transformations, will be denoted respectively by $A^c(x)$, $T_\lambda^c(x)$, and $T^c(x)$; for $i = (k,l)$ we put $\psi_i(x) = \psi_{kl}(x)$, $g_i(x) = g_{kl}(x)$.

Lemma 2. If

$$\sum_{k=1}^{n-1}\sum_{l=1}^{\infty} e^{n-1} \max_{0\le\alpha,\beta\le n-1} \left| e^{-\alpha-\beta}\varphi_{kl}^{(\alpha)}(0) f_{kl}^{(\beta)}(0)\right| < \infty \tag{16}$$

and $\det A^c(x) \neq 0$ $(0 \le x \le 1)$, then there exists an operator L_1—a $\{\varphi_{kl}, f_{kl}\}$-transformation of the operator L. The matrix $T_\lambda^c(x)$ satisfies relations of the form (7), (5), and (6), where

$$D_\nu^a = C_\nu^a [T^c(a)]^{-1}. \tag{17}$$

On the basis of Lemma 2, for a fixed system of forms $\{C_\nu^a\}$, by changing the normalization of the $\{C_\nu^a\}$-eigenfunctions φ_{kl} and f_{kl} $(1 \le k \le n-1;\ 1 \le l < \infty)$, we obtain an infinite set of $\{\varphi_{kl}, f_{kl}\}$-transformations of the operator L. The asymptotics of $\varphi_{kl}(x)$, $f_{kl}(x)$, λ_{kl}, and α_{kl} is determined by the methods of [8]. In particular, we obtain that

$$\alpha_{kl} = \varepsilon_k e^{-(n-2)} \left[1 + \sum_{m=1}^{n} h_{km}(L, \{C_\nu^a\}) e^{-m} + e^{-n} o_{kl} \right], \qquad \sum_{kl} |o_{kl}|^2 < \infty,$$

where ε_k are constants, and $h_{km}(L, \{C_\nu^a\})$ are expressed in finite form in terms of L and $\{C_\nu^a\}$.

Theorem 4. Let $L, L_1 \in \Lambda$; let $\{\lambda_{kl}, \alpha_{kl}\}$ and $\{\mu_{kl}, \beta_{kl}\}$ be respectively the $\{C_\nu^a\}$-spectral characteristic of the operator L and the $\{D_\nu^a\}$-spectral characteristic of the operator L_1, and suppose that condition a) is satisfied for the eigenvalues λ_{kl} and μ_{kl}. In order that the operator L_1 be a $\{\varphi_{kl}, f_{kl}\}$-transformation of the operator L for some normalization of the $\{C_\nu^a\}$-eigenfunctions $\varphi_{kl}(x)$ and $f_{kl}(x)$ $(1 \le k \le n-1;\ 1 \le l < \infty)$, and that (17) hold, it is necessary and sufficient that the relations

$$\lambda_{kl} = \mu_{kl}, \qquad h_{km}(L, \{C_\nu^a\}) = h_{km}(L_1, \{D_\nu^a\})$$

$$(1 \le k \le n-1; \quad 1 \le l < \infty; \quad 1 \le m \le n)$$

be satisfied.

There is the equality

$$\beta_{kl} = \frac{\alpha_{kl}}{1 + \int_0^1 \varphi_{kl}(t) \overline{f}_{kl}(t)\, dt} \qquad (1 \le k \le n-1, \quad 1 \le l < \infty).$$

The $\{D_\nu^a\}$-eigenfunctions of the operators L_1 and $\widetilde{L}_1$ are the functions $\psi_{kl}(x)$ and $g_{kl}(x)$.

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Note: Figure translations are in progress. See original paper for figures.