MATHEMATICS

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Abstract

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ALGEBRAIC THEORY OF GENERALIZED ANALYTIC FUNCTIONS ON CLOSED RIEMANN SURFACES

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Let $B(z)$ be a covariant, with respect to $\bar{z}$, continuous except for a finite number of points and lines of discontinuity, defined on a closed Riemann surface R of genus $p > 2$. The differential equation

$$U_{\bar{z}} = B(z)\,\overline{U} \tag{1}$$

is reduced to the integral equation

$$U(z) + \frac{1}{\pi}\iint_R B(t)\,\overline{U(t)}\,A(t,z)\,dT = \Phi(z), \tag{2}$$

where $\Phi(z)$ is an analytic function. Here $A(t, z)$ is the Cauchy kernel constructed in ([7]). Its properties have been studied in more detail in the author’ s paper ([3]). The kernel $A(t, z)$ has a pole with residue $+1$ when $P[t] = P[z]$ and poles of first order in z, determined by the divisor δ, $\mathrm{ord}(\delta) = p$, $\dim(W - \delta) = 0$, called characteristic. The principal parts of the expansions at the points δ are covariants of the first kind with respect to t, forming a basis. With respect to t, at the points δ the kernel has zeros and a pole of first order at the point P_0.

It is not difficult to introduce a metric in which the operator

$$KU \equiv -\frac{1}{\pi}\iint_R B(t)\,\overline{U(t)}\,A(t,z)\,dT \tag{3}$$

will be completely continuous.

The adjoint to (2) turns out to be the covariant equation

$$V(z) + \frac{1}{\pi}\iint_R B(t)\,\overline{V(t)}\,A(z,t)\,dT = \Psi(z), \tag{4}$$

where $\Psi(z)$ is an analytic covariant. Its solutions are covariants satisfying the differential equation

$$V_{\bar{z}} + \overline{B(z)}\,\overline{V} = 0, \tag{5}$$

which we shall call the adjoint of (1).

For the solvability of equation (2), it is necessary and sufficient that

$$\operatorname{Re} \iint_R \overline{B(t)}\,\overline{V_j(t)}\,\Phi(t)\,dt = 0 \quad (j = 1, 2, \dots, k), \tag{6}$$

where $V_j(z)$ $(j = 1, 2, \dots, k)$ is a complete system of solutions of the homogeneous equation (4).

For the solvability of equation (4), it is necessary and sufficient that

$$\operatorname{Re} \iint_R B(t)\,\overline{U_j(t)}\,\Psi(t)\,dT = 0 \quad (j = 1, 2, \dots, k), \tag{7}$$

where $U_j(z)$ $(j = 1, 2, \dots, k)$ is a complete system of solutions of the homogeneous equation (2).

As shown in ([3]), such solutions, generally speaking, exist.

From the integral representation

$$U(z) = \varphi(z) \exp\left\{ -\frac{1}{\pi} \iint_T B(t) \frac{\overline{U(t)}}{U(t)} A(t, z)\,dT \right\} \quad \text{in } T, \tag{8}$$

established by I. I. Danilyuk ([2]), the Liouville theorem easily follows in the following formulation: a solution of equation (2), regular everywhere on R and having at least one zero, is identically equal to zero. Hence it follows that the number k_0 of solutions of equation (1) regular on R (in what follows we call them generalized constants) does not exceed two. The number k_1 of covariants regular on R satisfying equation (5) (we call them generalized covariants of the first kind) does not exceed $2p$.

Examples show that the numbers k_0 and k_1 may vary depending on the choice of $B(z)$. In particular, for $B(z) \equiv 0$, $k_0 = 2$.

Let us construct an example with $k_0 < 2$. In the domain $R - T$ consider an analytic covariant having a pole of first order at P_0 and zeros in δ. Extend it continuously into T. Put $B(z) = -\overline{V_z(z)}/V(z)$, where $V(z)$ is the covariant constructed by us. Then it satisfies the integral equation

$$V(z)+\frac{1}{\pi}\iint_T \overline{B(t)}\,\overline{V(t)}\,A(z,t)\,dT=Z'(z), \tag{9}$$

where $Z'(z)$ is an Abelian covariant of the first kind having a zero in δ. As a consequence of the condition $\dim(W-\delta)=0$, $Z'(z)\equiv 0$. In view of the fact that the residue of $V(z)$ at the point P_0

$$-\frac{1}{\pi}\iint_T \overline{B(t)}\,\overline{V(t)}\,dT\neq 0, \tag{10}$$

at least one of the equations is insoluble,

$$U(z)+\frac{1}{\pi}\iint_T B(t)\overline{U(t)}\,A(t,z)\,dT=1\ (i). \tag{11}$$

It follows that $k_0<2$, since generalized constants, as is easy to see, satisfy one of these equations.

Theorem 1. *The difference between the number k_0 of generalized constants and the number k_1 of generalized covariants of the first kind is equal to*

$$k_1-k_0=2p-2. \tag{12}$$

Let $k_0=2$. Then both equations are soluble,

$$U(z)+\iint_R B(t)\overline{U(t)}\,A(t,z)\,dT=1\ (i), \tag{13}$$

and, consequently, for all solutions of the equation

$$V(z)+\frac{1}{\pi}\iint_R \overline{B(t)}\,\overline{V(t)}\,A(z,t)\,dT=0 \tag{14}$$

the condition

$$\iint_R \overline{B(t)}\,\overline{V(t)}\,dT=0 \tag{15}$$

is fulfilled. Thus all k solutions of equation (14) turn out to be generalized covariants of the first kind. Let us consider the solvability conditions for the equation

$$V(z)+\frac{1}{\pi}\iint_R \overline{B(t)}\,\overline{V(t)}\,A(z,t)\,dT=\sum_{i=1}^{2p}x_iZ_i'(z)^*, \tag{16}$$

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* $Z_i'(z)$ $(i = 1, 2, \dots, 2p)$

is a real basis of the Abelian differentials of the surface R.

among whose solutions all the remaining generalized covariants of the first kind are contained. We arrive at the system

$$\sum_{i=1}^{2p} x_i a_{ij} = 0 \qquad (j = 1, 2, \dots, k), \tag{17}$$

where

$$a_{ij} = \operatorname{Re} \iint_R B(t)\, \overline{U_j(t)}\, Z_i'(t)\, dT \qquad (i = 1, 2, \dots, k), \tag{18}$$

$U_j(t)$ $(j = 1, 2, \dots, k)$ is a complete system of solutions of the homogeneous equation (2). It is proved that rang $\|a_{ij}\| = k$, and, consequently, equation (16) has $2p-k$ linearly independent solutions, and all of them prove to be generalized covariants of the first kind. The total number of generalized covariants of the first kind is equal to $2p$, and formula (12) is valid.

When $k_0 = 1$ or $k_0 = 0$, among the solutions of equation (14) there are one or two having poles at P_0, and the number of generalized covariants of the first kind proves to be equal to $2p - 1$ or $2p - 2$. Formula (12) also remains valid in this case.

Theorem 2 (Riemann-Roch). *The difference between the number M of functions satisfying equation (1) and divisible by the divisor $-\Delta$, and the number N of covariants satisfying equation (5) and divisible by Δ, is equal to*

$$M - N = 2 \operatorname{ord}(\Delta) - 2p + 2. \tag{19}$$

Divide R into domains T^+ and T^- so that all points of Δ lie in T^-. We have the representations

$$U(z) = U^+(z) \qquad \text{in } T^+,$$
$$U(z) = \Delta(z) U^-(z) \quad \text{in } T^-,$$

where $\Delta(z)$ is a function analytic in T^- whose poles are determined by the divisor Δ; $U^{\pm}(z)$ are functions regular in $T^{\pm}$ and satisfying the equation $U_{\bar z} = B_1(z)\overline{U}$, where

$$B_1(z) = \begin{cases} B(z), & \text{in } T^+, \\ B(z)\dfrac{\overline{\Delta(z)}}{\Delta(z)}, & \text{in } T^-. \end{cases} \tag{20}$$

On the contour Γ we arrive at the Riemann problem

$$U^+(t) = \Delta(t)U^-(t) \tag{21}$$

of index $\chi = \operatorname{ord}(\Delta)$.

For covariants we obtain the problem

$$V^+(t) = \frac{1}{\Delta(t)}V^-(t) \tag{22}$$

for the conjugate equation.

The assertion of the theorem follows from the results of work ([5]).

In conclusion, consider the equation

$$\partial_{\bar z}\omega_\nu = A\omega_\nu + B\overline{\omega_\nu} \tag{23}$$

for differentials of dimension ν.

Conjugate to the Riemann problem

$$\omega_\nu^+(t) = G(t)\omega_\nu^-(t) + g_\nu(t), \qquad (\omega_\nu) + \Delta \geqslant 0, \tag{24}$$

for differentials belonging to equation (23) and divisible by $-\Delta$ (see ([4])), is called the problem

$$\eta_{1-\nu}^+(t) = \frac{1}{G(t)}\eta_{1-\nu}^-(t), \qquad (\eta_{1-\nu}) - \Delta \geqslant 0, \tag{25}$$

for differentials of dimension $1-\nu$, multiples of Δ, and satisfying the equation

$$\partial_z\eta_{1-\nu} + A\eta_{1-\nu} + \overline{B}\,\overline{\eta}_{1-\nu} = 0, \tag{26}$$

conjugate to (23).

Using the results of work [6], one can prove the following assertions:

Theorem 3. *The difference between the number l of solutions of the homogeneous problem (24) and the number l' of solutions of the conjugate problem (25) is equal to*

$$l - l' = 2\varkappa + (2\nu - 1)(2p - 2) + 2\ \mathrm{ord}(\Delta), \qquad \varkappa = \mathrm{ind}_\Gamma G. \tag{27}$$

For solvability of the nonhomogeneous problem it is necessary and sufficient that

$$\mathrm{Im} \int_\Gamma g_\nu(\tau)\eta_{1-\nu}^{+(j)}(\tau) = 0 \qquad (j = 1, 2, \dots, l'), \tag{28}$$

where $\eta_{1-\nu}^{+(j)}$ $(j = 1, 2, \dots, l')$ is a complete system of solutions of problem (25).

Theorem 4. The difference between the number M of differentials of dimension ν, multiples of $-\Delta$ and satisfying equation (23), and the number N of differentials of dimension $1 - \nu$, satisfying the conjugate equation (26) and multiples of Δ, is equal to

$$M - N = 2\ \mathrm{ord}(\Delta) + (2\nu - 1)(2p - 2). \tag{29}$$

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Note: Figure translations are in progress. See original paper for figures.