# MATHEMATICS

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## Abstract

## Full Text

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# REPRESENTATIONS OF COMPACT RING GROUPS

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1. In the note [1], ring groups were introduced as a generalization of unimodular locally compact groups. Each ring group $\mathfrak{G}$ is determined by a Hilbert space $\mathfrak{H}$, a standard ring of operators $\mathfrak{M}$ in the space $\mathfrak{H}$ (the carrier of $\mathfrak{G}$), a fixed measure $m$ on the ring $\mathfrak{M}$ (an invariant measure), and two mappings: a determining isomorphism $\Phi : \mathfrak{M} \to \mathfrak{M} \otimes \mathfrak{M}$ and an involution $+ :\ A \to A^{+}$ $(A, A^{+} \in \mathfrak{M})$, satisfying certain conditions corresponding to the group axioms. In the present note it is shown that many propositions and methods of the representation theory of locally compact groups can be extended to ring groups. Mainly the simplest case of representations of ring groups, which is an analogue of compact ones, is considered.

2. A ring group is called **compact** if the identity operator of the carrier is summable with respect to the invariant measure $(m(I) < \infty)$. A ring group is called **discrete** if its carrier is a direct sum of factors of class $I_n$

$$\mathfrak{M} = \mathfrak{M}_1 \oplus \mathfrak{M}_2 \oplus \mathfrak{M}_3 \oplus \cdots \oplus \mathfrak{M}_m \oplus \cdots . \tag{1}$$

Each factor $\mathfrak{M}_m$ consists of matrices of the form

$$\begin{array}{|c|c|c}\hline |A| & & \\ \hline & |A| & \\ \hline & & \ddots \end{array} \qquad (M_m \text{ blocks}), \tag{2}$$

where $A$ runs through all matrices of order $M_m$. A ring group is called **finite** if $\mathfrak{H}$ is a finite-dimensional space. These definitions agree with the traditional ones for the case when $\mathfrak{G}$ is a locally compact group. The following theorem transfers to ring groups a well-known result of L. S. Pontryagin.

**Theorem 1.** *The ring group $\widehat{\mathfrak{G}}$, dual to a compact ring group $\mathfrak{G}$, is discrete. The ring group $\widehat{\mathfrak{G}}$, dual to a discrete ring group $\mathfrak{G}$, is compact.*

Recall that for a ring group a convolution operation $*$ can be defined, with respect to which (and the involution $+$) the summable operators form a Banach ring with involution $L_1$—the group algebra [1].

**Theorem 2.** *A ring group $\mathfrak{G}$ is discrete if and only if its group algebra has an identity $\delta$. The operator $\delta \in \mathfrak{M}$ is proportional to a one-dimensional central projection of the carrier, and $m(\delta) = 1$.*

Thus, in (1) one of the factors is the ring of one-dimensional matrices.

**Theorem 3.** *Up to a positive factor, the invariant measure on a discrete ring group is computed by the formula*

$$m(A) = \mathrm{Sp}(A) \qquad (A \in L_1(\mathfrak{M})). \tag{3}$$

From (1) and (3) it follows immediately that a ring group is finite if and only if it is discrete and compact.

3. It is known that every representation of a locally compact group can naturally be extended to a representation of its group algebra.

and that all representations of the group algebra can be obtained in this way ([2]). This justifies the following definition. A representation of a ring group is a representation of its group algebra. Thus, if $D(A)$ $(A \in L_1)$ is a representation of $\mathfrak{G}$ by operators in the Hilbert space $\mathfrak{R}$, then

$$D(A * B) = D(A)D(B), \qquad D(A^+) = D(A)^* \quad (A, B \in L_1). \tag{4}$$

From the definition of convolution it follows that the operators $X \to A * X$ $(X \in L_1(\mathfrak{M}) \cap L_2(\mathfrak{M}))$ are bounded in $L_2(\mathfrak{M})$ for every $A \in L_1(\mathfrak{M})$. Extending them by continuity to all of $L_2(\mathfrak{M})$, we obtain a representation of the ring group $\mathfrak{G}$. We shall call it the regular representation of $\mathfrak{G}$.

Using the isometry between $L_2(\mathfrak{M})$ and $L_2(\hat{\mathfrak{M}})$ generated by the Fourier transform ([1]), it is not difficult to show that the Neumann ring of the regular representation (i.e. the Neumann ring of operators generated by the operators of the representation) is spatially isomorphic to the carrier $\hat{\mathfrak{M}}$ of the dual group. From Theorem 1 and the definition of a discrete ring group it follows that the regular representation of a compact ring group decomposes into a direct sum of finite-dimensional irreducible representations.

Let $D(A)$ be a finite-dimensional representation of the ring group $\mathfrak{G}$. Choose an orthonormal basis $\{\xi_i\}$ in the representation space. The matrix elements $(D(A)\xi_j, \xi_i)$ are continuous functionals on $L_1(\mathfrak{M})$. In accordance with the general form of functionals of the space $L_1$ ([3]), there exists an operator $Z_{ij} \in \mathfrak{M}$ such that

$$(D(A)\xi_j, \xi_i) = m(AZ_{ij}). \tag{5}$$

We shall call the operators $Z_{ij}$ the coefficients of the representation $D$ (in the given basis). In the case when $\mathfrak{G}$ is an ordinary group, the coefficients $Z_{ij}$ are the matrix elements of the representation, regarded as functions on the group. From (4) and the definition of convolution it follows that the operators $Z_{ij}$ $(i, j = 1, 2, \dots, N)$ of the carrier are the coefficients of some representation if and only if

$$Z_{ij}^{+} = Z_{ji}, \qquad \Phi[Z_{ij}](1,2) = \sum_{s=1}^{N} Z_{is}(1) Z_{sj}(2), \qquad \sum_{s} Z_{si} Z_{sj}^{*} = \delta_{ij} I. \quad (6)$$

If $D(A)$ is a representation, then $D^{*}(A) = \overline{D(A^{*})}$ is also a representation, which we shall call conjugate to $D$. The coefficients of the conjugate representation are the operators $Z_{ji}^{*}$. Let $D'$ and $D''$ be two finite-dimensional representations with coefficients $Z'_{ij}$ $(i, j = 1, 2, \dots, N')$ and $Z''_{kl}$ $(k, l = 1, 2, \dots, N'')$, respectively. The operators $Z_{ik,jl} = Z'_{ij} Z''_{kl}$ satisfy the equalities (6) and therefore are the coefficients of some representation of dimension $N'N''$. We shall call it the Kronecker product of the representation $D'$ by $D''$: $D' \times D''$. It can be shown that the representation $D' \times D''$ does not depend on the choice of bases in which the coefficients $Z'_{ij}$ and $Z''_{kl}$ were computed. When $\mathfrak{G}$ is an ordinary group, $D' \times D''$ reduces to the Kronecker product of the representations $D'$ and $D''$ in the usual sense. In contrast to this classical case, the operation of the Kronecker product is, generally speaking, noncommutative. The relations

$$(D' + D'') \times D''' = D' \times D''' + D'' \times D''', \qquad D' \times (D'' + D''') = D' \times D'' + D' \times D''',$$

$$(D^{*})^{*} = D, \qquad (D' + D'')^{*} = {D'}^{*} + D''^{*}, \qquad (D' \times D'')^{*} = D''^{*} \times {D'}^{*}$$

make it possible, in the usual way, to define the ring of representations (noncommutative, with involution $*$). By analogy with groups, we shall call the operator

$$\chi = \sum_{i=1}^{N} Z_{ii}$$

the character of the finite-dimensional representation $D$. The character

determines the representation $D$ uniquely up to unitary equivalence. From (6) it follows that $\chi^{+} = \chi$, $\Phi[\chi] = \widetilde{\Phi}[\chi]$. The representations $D^{*}$, $D' + D''$ and $D' \times D''$ have respectively the characters $\chi^{*}$, $\chi' + \chi''$, $\chi'\chi''$.

In the works of Tannaka ([4]) and M. G. Krein ([5]) for the case of a compact group, and in the work of Stinespring ([6]) for the case of an arbitrary unimodular group, L. S. Pontryagin' s construction of a group $G$ from its dual object (the dual ring group) was generalized. This construction can be formulated in the following way, using Kronecker products of representations. Let $G$ be a unimodular group. Its dual ring group $G$ is commutative ([1]), and therefore all its irreducible representations are one-dimensional. Denote by $G_1$ the set of all irreducible representations of the ring group $\widehat{G}$. Introduce in $G_1$ the operation of multiplication by setting the product of an element $D'$ by $D''$ equal to $D' \times D''$, and introduce in $G_1$ a topology by taking as neighborhoods of an element $D$ the set of all $D'$ satisfying the inequalities $|D(A_i) - D'(A_i)| < \varepsilon_i$, where $\{A_i\}$ is an arbitrary fixed finite set of operators from $L_1(\mathfrak{M})$, $\varepsilon_i > 0$. With respect to the multiplication operation and topology thus introduced, $G_1$ forms a locally compact group isomorphic to the group $G$.

4. Let $\mathfrak{G}$ be a compact ring group. Denote by $D_n$ the irreducible representations of $\mathfrak{G}$ occurring in its regular representation. Let $N_n$ be the dimension of the representation $D_n$; let $Z_{ij,n}$ $(i, j = 1, 2, \dots, N_n)$ be its coefficients (computed in an arbitrary fixed basis). The invariant measure of the group is normalized by the condition $m(I) = 1$.

**Theorem 4.** An arbitrary representation of a compact ring group is a direct sum of its irreducible representations occurring in the regular representation. The coefficients of these irreducible representations $Z_{ij,n}$ form a complete orthogonal system in the space $L_2(\mathfrak{M})$. They are normalized so that $\|Z_{ij,n}\|_2 = N_n^{-1/2}$. The equalities $Z_{ij,n} * Z_{kl,m} = N_n^{-1} Z_{il,n}$ hold if $m = n,\ j = k$, and it is equal to zero otherwise.

From this theorem there follow in the usual way, for compact ring groups, the basic propositions of the representation theory of compact groups.

5. Let $\mathfrak{G}$ be an arbitrary ring group. Each operator $Z \in \mathfrak{M}$ defines on $L_1$ the functional $m(AZ)$. We shall call an operator $Z \neq 0$ positive definite if this functional: a) is positive (i.e. $m((C^+ * C)Z) \geq 0$ for all $C \in L_1(\mathfrak{M})$); b) can be extended to a positive functional on the algebra $L_1$ supplemented by a formal unit. We shall call an operator $Z$ normalized if the functional takes the value 1 on the unit. Formula (5) associates with each representation $D(A)$ and representation vector $\xi$ a certain positive definite operator $Z$ (put $\xi = \xi_i = \xi_j,\ Z_{ij} = Z$). Moreover, the known construction ([2]) makes it possible, from any positive definite operator $Z$, to construct a representation $D(A)$ and a vector $\xi$ related to it by equality (5). As in the case of groups, in this way we obtain a one-to-one correspondence between positive definite operators $Z$ and cyclic representations $D(A)$ with cyclic vectors $\xi$. A standard group-theoretic argument ([7]) shows that any normalized positive definite operator $Z$ can be represented in the form

$$Z = \int Z_\lambda \, d\nu(\lambda), \tag{7}$$

where all $Z_\lambda$ are normalized positive definite elementary operators (i.e. corresponding to irreducible representations), and $\nu$ is a finite measure ($\int d\nu(\lambda) = 1$). The integral is understood in the weak sense.

(as in the space conjugate to $L_1$). This assertion reduces to Bochner' s theorem in the case when $\mathfrak{G}$ is a discrete group. For an arbitrary unimodular group it is somewhat weaker than Bochner' s theorem (in Bochner' s theorem the integral (7) converges at every element of the group). The basic results and methods of Godement' s character theory ([8]) can also be carried over to ring groups.

**Remark.** In the case when $\mathfrak{G}$ is an ordinary group, the definition of a positive definite operator coincides with the definition of an integrable positive definite function. In this case condition b) turns out to be superfluous; it follows from a). Condition b) is also superfluous in the case when $\mathfrak{G}$ is a discrete or compact ring group. Whether condition b) follows from a) in the case of an arbitrary ring group is unknown.

One of the main differences between the representation theory of groups (and, consequently, of group algebras) and the representation theory of arbitrary semisimple Banach rings with involution is that for the former one can define the Kronecker product of representations. The construction by which, in Section 3, the Kronecker product of finite-dimensional representations was introduced admits a generalization to certain infinite-dimensional representations. However, the question of the possibility of defining the notion of the Kronecker product for any pair of infinite-dimensional representations of a ring group remains open.

6. In conclusion we describe, by means of irreducible representations, for the case of a finite ring group $\mathfrak{G}$, the construction of the dual group $\widehat{\mathfrak{G}}$. Let us first note that a finite ring group is uniquely determined by specifying the carrier $\mathfrak{M}$ and a set $Z_{ij,n}$ of coefficients of the irreducible representations. Indeed, the operators $Z_{ij,n}$ form a basis in $\mathfrak{M}$, and therefore the equalities (6) uniquely determine the defining isomorphism $\Phi$ and the involution $+$ of the ring group. It will be convenient for us to define the invariant measure by the equality $m(A) = N^{-1/2}\,\mathrm{Sp}(A)$, where $N$ is the dimension of $\mathfrak{H}$ (cf. ([3])). Choose in each $\mathfrak{M}_m$ (see ([1], [2])) a basis consisting of matrices $T_{kl,m}$, for which the corresponding matrix $A$ has zeros everywhere except for the element in the $k$-th row and $l$-th column, equal to

$$\frac{\sqrt{N}}{M_m}.$$

Let us note that, from the prescribed set of numbers $\{M_m\}$, the ring $\mathfrak{M}$ is uniquely determined. The operators $M_m^{1/2} N^{-1/4} T_{kl,m}$ form a complete orthonormal system in $L_2(\mathfrak{M})$. According to Theorem 4 (taking into account the new normalization of the measure), $N_n^{1/2} N^{-1/4} Z_{ij,n}$ is also a complete orthonormal system. Denote by $\Omega$ the unitary matrix connecting the bases $\{N_n^{1/2} N^{-1/4} Z_{ij,n}\}$ and $\{M_m^{1/2} N^{-1/4} T_{ij,m}\}$:

$$\{N_n^{1/2} N^{-1/4} Z_{ij,n}\} = \Omega\{M_m^{1/2} N^{-1/4} T_{ij,m}\}.$$

**Theorem 5.** *Put* $\widehat{M}_n = N_n$, $\widehat{N}_m = M_m$, *and, from the set of numbers* $\{\widehat{M}_n\}$, *construct the ring* $\widehat{\mathfrak{M}}$ *and the operators* $T$, *which we denote by* $T^*_{ij,n}$. *Then, with the aid of the matrix* $\Omega^{-1}$, *define the operators* $Z^*_{kl,m}\{\widehat{N}_m^{1/2} N^{-1/4} Z^*_{kl,m}\} = \Omega^{-1}\{\widehat{M}_n^{1/2} N^{-1/4} T^*_{ij,n}\}$. *The ring* $\widehat{\mathfrak{M}}$ *is the carrier of the dual group* $\widehat{\mathfrak{G}}$, *and the operators* $Z^*_{kl,m}$ *are the coefficients of its irreducible representations.*

The Fourier transform of the operator $T_{kl,m}$ is the operator $Z^*_{kl,m}$. This uniquely determines the Fourier transform for the ring group.

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*Note: Figure translations are in progress. See original paper for figures.*