A SUPERCONDUCTIVITY CRITERION AS A CRITERION FOR THE APPEARANCE OF AN INSTABILITY OF THE BETHE–SALPETER EQUATION FOR THE NORMAL STATE

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Abstract

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A SUPERCONDUCTIVITY CRITERION AS A CRITERION FOR THE APPEARANCE OF AN INSTABILITY OF THE BETHE-SALPETER EQUATION FOR THE NORMAL STATE

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Earlier (^{1}) we obtained an exact form of the superconductivity criterion on the basis of a variant of the theory of superconductivity in which extraneous sources are introduced that create and annihilate superconducting pairs of electrons. In this case the superconductivity criterion was obtained as a condition on the one-particle Green' s function of the normal state and the Schwinger interaction operator of the normal state. Thus we obtained a superconductivity criterion from the side of the superconducting state. We shall now obtain this criterion from the side of the normal state by studying the appearance of a special instability in the Bethe-Salpeter equation for the normal state. In essence, the present work is a refined version of Cooper' s original work (^{2}).

The equation for the wave function of a two-electron bound state with zero total spin for an electron-phonon system with Coulomb interaction in the momentum representation has the form

$$\psi(k_1t_1, k_2t_2) = -\frac{i}{\hbar}\frac{1}{V}\int_{-\infty}^{+\infty} d\tau_1 \int_{-\infty}^{+\infty} d\tau_2 \int_{-\infty}^{+\infty} d\tau_1' \sum_{k_1'} \int_{-\infty}^{+\infty} d\tau_2' \sum_{k_2'} G(k_1, t_1-\tau_1) \times G(k_2, t_2-\tau_2) I(k_1\tau_1, k_2\tau_2; k_1'\tau_1', k_2'\tau_2') \psi(k_1'\tau_1', k_2'\tau_2'), \tag{1}$$

where ψ is the wave function of the bound state; I is the Schwinger interaction operator for zero total spin; G is the Green one-particle function; k is the wave vector.

In the energy representation we have

$$\psi(k_1E_1, k_2E_2) = -\frac{i}{\hbar}\frac{1}{V}\frac{1}{(2\pi\hbar)^2}\int_{-\infty}^{+\infty} dE_1' \int_{-\infty}^{+\infty} dE_2' \sum_{k_1'}\sum_{k_2'} G(k_1E_1)G(k_2E_2) \times I(k_1E_1, k_2E_2; k_1'E_1', k_2'E_2')\psi(k_1'E_1', k_2'E_2'), \tag{2}$$

whence, passing to relative coordinates,

$$\frac{1}{2}(E_2 - E_1) = E, \qquad \frac{1}{2}(E_2' - E_1') = E', \qquad E_1 + E_2 = E_1' + E_2' = \mathcal{E},$$

$$\frac{1}{2}(k_2 - k_1) = k, \qquad \frac{1}{2}(k_2' - k_1') = k', \qquad k_1 + k_2 = k_1' + k_2' = \mathcal{K},$$

we obtain

$$\psi(kE; \mathcal{K}\mathcal{E}) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} dE' \,\frac{1}{V}\sum_{k'}\left(\frac{i}{\hbar}\right)^2 G\left(\frac{\mathcal{K}}{2} + k, \frac{\mathcal{E}}{2} + E\right) \times G\left(\frac{\mathcal{K}}{2} - k, \frac{\mathcal{E}}{2} - E\right) I(kE, k'E'; \mathcal{K}\mathcal{E})\psi(k'E', \mathcal{K}\mathcal{E}),$$

where $I(kE, k'E'; \mathcal{K}\mathcal{E})$ for $\mathcal{K} = 0$ and $\mathcal{E} = 0$ is given by the expression

$$I(kE, k'E') = \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} dt_2 \, e^{\frac{i}{\hbar}Et - \frac{i}{\hbar}E'(t_2 - t_1)} I(-k, 0; k, t; -k', t_1; k', t_2). \tag{3}$$

which coincides with the corresponding operator entering the original equation (8) of (1).

With the aid of the equations given, let us examine in more detail the question of a bound state, for which $\mathcal{K} = 0$ and $\mathcal{E}$ is sufficiently small. In this case one may retain the dependence on $\mathcal{E}$ only in G, and neglect it in I. In addition, we make the substitution of the unknown function

$$\varphi(kE; \mathcal{K}\mathcal{E}) = G^{-1}\left(\frac{\mathcal{K}}{2} + k, \frac{\mathcal{E}}{2} + E\right) G^{-1}\left(\frac{\mathcal{K}}{2} - k, \frac{\mathcal{E}}{2} - E\right)\psi(kE; \mathcal{K}\mathcal{E})$$

and pass from the sum to an integral

$$\frac{1}{V}\sum_{k'} \ldots = \frac{1}{2\pi^2}\int_0^{+\infty} {k'}^2 \, dk' \ldots ,$$

simultaneously replacing $I(kE, k'E')$ by the angular average $K(kE, k'E')$. Then we arrive at the necessity of studying the equation

$$\varphi(kE) = \frac{i}{4\pi^3} \int_0^{+\infty} {k'}^2\, dk' \int_{-\infty}^{+\infty} dE'\, K(kE, k'E') \left(\frac{i}{\hbar}\right)^2 G\left(k', \frac{\mathcal{E}}{2} + E'\right) \times$$

$$\times G\left(k', \frac{\mathcal{E}}{2} - E'\right) \varphi(k'E'). \tag{4}$$

Let us find the leading terms of the asymptotic expansion of the right-hand side of (4) for small $|\mathcal{E}| = -\mathcal{E}$. We shall use here the fact that near the Fermi surface

$$G(kE) \simeq \frac{\hbar}{i} \frac{Z(k)}{-E + E(k) - \mu - i\Gamma(k)} \quad \text{as } |k| \to k_F.$$

The Fermi wave vector is determined from the condition $E(k_F) - \mu = 0$. The damping $\Gamma(k)$ for $|k| < k_F$ is negative, for $|k| > k_F$ positive; for $|k| = k_F$ it vanishes; moreover, $|\Gamma(k)| \ll |E(k) - \mu|$ as $|k| \to k_F$.

We divide the whole region of integration over k' into two regions, the outer and the inner (the outer region from 0 to $k_F - \Delta$ and from $k_F + \Delta$ to $+\infty$, the inner from $k_F - \Delta$ to $k_F + \Delta$). Below we shall show that the integral over the inner region gives, asymptotically as $|\mathcal{E}| \to 0$, logarithmic terms $\ln|\mathcal{E}| + \text{const} + \ldots$, while the integral over the outer region gives, asymptotically as $|\mathcal{E}| \to 0$, weaker terms const + ...

Consider the integral over the outer region. For it we have

$$\frac{i}{4\pi^3} \left(\int_0^{k_F - \Delta} {k'}^2\, dk' + \int_{k_F + \Delta}^{+\infty} {k'}^2\, dk' \right) \int_{-\infty}^{+\infty} dE'\, K(kE, k'E') \times$$

$$\times \left(\frac{i}{\hbar}\right)^2 G\left(k', \frac{\mathcal{E}}{2} + E'\right) G\left(k', \frac{\mathcal{E}}{2} - E'\right) \varphi(k'E') \simeq$$

$$\simeq \frac{i}{4\pi^3} \left(\int_0^{k_F - \Delta} {k'}^2\, dk' + \int_{k_F + \Delta}^{+\infty} {k'}^2\, dk' \right) \int_{-\infty}^{+\infty} dE'\, K(kE, k'E') \times$$

$$\times \left(\frac{i}{\hbar}\right)^2 G(k'E') G(k', -E') \varphi(k'E') \quad \text{as } |\mathcal{E}| \to 0. \tag{5}$$

We have neglected all dependence on $|\mathcal{E}|$. The resulting integral converges because of the presence of Δ. It is convenient to take the splitting parameter Δ

sufficiently small. Therefore for (5) we should take the asymptotic formula, for small Δ,

$$\frac{i}{(4\pi)^3}\left(\int_0^{k_F+\Delta} {k'}^2\,dk' + \int_{k_F+\Delta}^{+\infty} {k'}^2\,dk'\right)\int_{-\infty}^{+\infty} dE'\,K(kE,k'E')\times$$

$$\times\left(\frac{i}{\hbar}\right)^2 G(k'E')G(k',-E')\varphi(k'E') \simeq \frac{1}{2\pi^2}\frac{k_F^2 Z^2(k_F)}{E'(k_F)}K(kE,k_F0)\varphi(k_F0)\ln\Delta -$$

$$-\frac{i}{4\pi^3}\int_0^{+\infty} dk' \ln|k'-k_F|\frac{d}{dk'}\left({k'}^2(k'-k_F)\int_{-\infty}^{+\infty} dE' K(kE,k'E')\times\right.$$

$$\left.\times\left(\frac{i}{\hbar}\right)^2 G(k'E')G(k',-E')\varphi(k'E')\right) \quad \text{as } \Delta\to 0. \tag{6}$$

Substituting (6) into (5), we obtain the final asymptotic expression, for small $|\mathcal{E}|$ and small Δ, for the contribution from the outer region of integration.

Let us consider the integral over the inner region. For it, for $G(k',\mathcal{E}/2+E')$ and $G(k',\mathcal{E}/2-E')$, we may use the asymptotic expressions for the immediate vicinity of $k'=k_F$, so that

$$\frac{1}{4\pi^3}\int_{k_F-\Delta}^{k_F+\Delta} {k'}^2 dk' \int_{-\infty}^{+\infty} dE' K(kE,k'E')\left(\frac{i}{\hbar}\right)^2 G\left(k',\frac{\mathcal{E}}{2}+E'\right)\times$$

$$\times G\left(k',\frac{\mathcal{E}}{2}-E'\right)\varphi(k'E') \simeq \frac{i}{4\pi^3}\int_{k_F-\Delta}^{k_F+\Delta} {k'}^2 dk' \int_{-\infty}^{+\infty} dE' K(kE,k'E')\times$$

$$\times\frac{Z^2(k')\varphi(k'E')}{-{E'}^2+\left(-\mathcal{E}/2+E(k')-\mu-i\Gamma(k')\right)^2} \quad \text{as } \Delta\to 0. \tag{7}$$

We make a stretching of the region of integration

$$k' = k_F + \frac{|\mathcal{E}|}{2E'(k_F)}x, \qquad E' = \frac{|\mathcal{E}|}{2}y,$$

in order to exhibit the contribution to the asymptotics from the immediate vicinity of $k'=k_F$ and $E'=0$. Then we shall have

$$\frac{i}{4\pi^3}\int_{k_F-\Delta}^{k_F+\Delta} {k'}^2 dk' \int_{-\infty}^{+\infty} dE' K(kE,k'E')\frac{Z^2(k')\varphi(k'E')}{-{E'}^2+\left(-\frac{\mathcal{E}}{2}+E(k')-\mu-i\Gamma(k')\right)^2} \simeq$$

$$\simeq \frac{i}{4\pi^2}\frac{k_F^2 Z^2(k_F)}{E'(k_F)}K(kE,k_F0)\varphi(k_F0)\int_{-\frac{2\Delta E'(k_F)}{|\mathcal{E}|}}^{\frac{2\Delta E'(k_F)}{|\mathcal{E}|}} dx \int_{-\infty}^{+\infty} dy\, \frac{1}{-y^2+(1+x-i\varepsilon\,\mathrm{sign}\,x)^2} =$$

$$= -\frac{1}{4\pi^2}\frac{k_F^2 Z^2(k_F)}{E'(k_F)}K(kE,k_F0)\varphi(k_F0)\ln\left(-\frac{4\Delta^2(E'(k_F))^2}{|\mathcal{E}|^2}\right) \quad \text{as } |\mathcal{E}|\to 0. \quad (8)$$

We integrated first with respect to y, closing the contour in the upper half-plane, and then with respect to x; ε denotes an infinitely small positive number; the logarithmic function of a negative number is understood in the sense $\ln(-x) = \ln x + i\pi$.

Substituting (8) into (7), we obtain the final asymptotic expression, for small Δ and small $|\mathcal{E}|$, for the contribution from the inner region of integration.

Putting together (5), (6), (7), (8), we obtain the required leading terms of the asymptotic expansion of the right-hand side of (4) for small $|\mathcal{E}|$:

$$\frac{i}{4\pi^3}\int_0^{+\infty} {k'}^2\,dk' \int_{-\infty}^{+\infty} dE'\,K(kE,k'E')\left(\frac{i}{\hbar}\right)^2 G\left(k',\frac{\mathcal{E}}{2}+E'\right)G\left(k',\frac{\mathcal{E}}{2}-E'\right)\varphi(k'E') \simeq$$

$$\simeq -\frac{1}{4\pi^2}\frac{k_F^2 Z^2(k_F)}{E'(k_F)}K(kE,k_F0)\varphi(k_F0)\ln\left(-\frac{4(E'(k_F))^2}{|\mathcal{E}|^2}\right) -$$

$$-\frac{i}{4\pi^3}\int_0^{+\infty} dk'\;\ln|k'-k_F|\,\frac{d}{dk'}\Big({k'}^2(k'-k_F)\times$$

$$\times\int_{-\infty}^{+\infty} dE'\,K(kE,k'E')\left(\frac{i}{\hbar}\right)^2 G(k'E')G(k',-E')\varphi(k'E')\Big) \quad \text{as } |\mathcal{E}|\to 0. \quad (9)$$

As was to be expected, the terms with Δ have dropped out of (9), since in the final formula nothing should depend on the splitting parameter. Substituting (9) into (4), we obtain the asymptotic form of equation (4)

$$\begin{aligned}\varphi(kE) = &-\frac{1}{4\pi^2}\frac{k_F^2 Z^2(k_F)}{E'(k_F)}K(kE,k_F0)\varphi(k_F0)\ln\left(-\frac{4\omega^2 Z^2(k_F)}{|\mathcal{E}|^2}\right)\\ &-\frac{i}{4\pi^3}\int_0^{+\infty} dk'\;\ln\frac{E'(k_F)|k'-k_F|}{\omega Z(k_F)}\,\frac{d}{dk'}\Big({k'}^2(k'-k_F)\times\\ &\times\int_{-\infty}^{+\infty} dE'\;K(kE,k'E')\left(\frac{i}{\hbar}\right)^2 G(k'E')G(k',-E')\varphi(k'E')\Big),\end{aligned} \quad (10)$$

in which, for convenience of dimensional considerations, we have introduced an arbitrary quantity ω of the dimension of energy.

Let us introduce a new unknown function

$$f(kE) = \frac{\varphi(kE)}{-\varphi(k_F0)\,{}^{1}/_{2}\ln\left(-4\omega^2 Z^2(k_F)/|\mathcal{E}|^2\right)}, \tag{11}$$

whence, putting $k = k_F$, we have

$$|\mathcal{E}|^2 = -4\omega^2 Z^2(k_F)e^{2/f(k_F0)}. \tag{12}$$

For $f(kE)$, from (10) we obtain

$$\begin{aligned} f(kE) = &\frac{1}{2\pi^2}\frac{k_F^2 Z^2(k_F)}{E'(k_F)}K(kE, k_F0) \\ &- \frac{i}{4\pi^3}\int_0^{+\infty} dk'\ \ln\frac{E'(k_F)|k'-k_F|}{\omega Z(k_F)}\frac{d}{dk'}\Big({k'}^2(k'-k_F)\times \\ &\times\int_{-\infty}^{+\infty} dE'\ K(kE, k'E')\left(\frac{i}{\hbar}\right)^2 G(k'E')G(k', -E')f(k'E')\Big). \end{aligned} \tag{13}$$

Equation (13) coincides exactly with equation (12) from (1).

We shall now formulate the instability criterion. According to (12), in order to ensure the smallness of $|\mathcal{E}|$, it is necessary and sufficient to require that $f(k_F0)$ be negative and small (to solve (13), one should put in it $|\mathcal{E}|$ equal to a purely imaginary number, so that the equation for the two-electron bound state has no small real eigenvalue, which we have been seeking up to now). Thus, the critical form of the interaction for which the instability effect with respect to the formation of superconducting pairs of electrons appears is determined from the condition

$$f(k_F0) = 0. \tag{14}$$

The instability criterion obtained on the basis of the Bethe–Salpeter equation coincides exactly with the superconductivity criterion from (1). The coincidence of these criteria, obtained as a result of completely different approaches, reveals the internal consistency of the theory of superconductivity.

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Note: Figure translations are in progress. See original paper for figures.

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