ON THE EXCITATION OF MAGNETOSONIC WAVES IN A CONDUCTING FLUID

1962

SovietRxiv

Abstract

Full Text

HYDROMECHANICS

S. V. IORDANSKII

ON THE EXCITATION OF MAGNETOSONIC WAVES IN A CONDUCTING FLUID

(Presented by Academician L. I. Sedov, 11 V 1962)

In a number of works ([1]-[3]) it is proposed to use the resonant excitation of magnetosonic oscillations in a bounded volume of plasma for the purpose of heating it. In this case, to calculate the heating rate, a linear theory is used, according to which the amplitude of the oscillations that arise grows without bound as the dissipation decreases. However, for a fixed amplitude of the external field, the relative role of the nonlinear terms increases as the magnitude of the dissipative coefficients decreases, despite the fact that the amplitude of the oscillations itself may still remain small.

In the present article the excitation of one-dimensional magnetosonic oscillations in a bounded volume of a conducting fluid is considered in the approximation of magnetic hydrodynamics with infinite conductivity. The case is studied in which direct dissipative processes may be neglected in comparison with nonlinear effects.

As is known, the linearized equations of magnetic hydrodynamics admit three types of independent solutions ([5]): fast and slow magnetosonic waves and Alfvén waves. To each of the waves there correspond two arbitrary functions

$$\Lambda_{1,2}^{\pm} = \frac{H_{0y}}{u_{1,2}^2 - c_0^2}\frac{c_0^2}{\rho_0}\rho' \pm \frac{H_{0y}}{u_{1,2}^2 - c_0^2} u_1 v_x + \frac{H_{0x}}{u_{1,2}} v_y + h_y, \qquad \Lambda_3^{\pm} = \left[v_z \mp \frac{h_z}{\sqrt{4\pi\rho_0}} \right],$$

where H_{0y}, H_{0x} are components of the stationary magnetic field; h_y, h_z are components of the perturbation of the magnetic-field vector; v_x, v_y, v_z are components of the velocity vector; ρ_0 is the unperturbed density, ρ' the density perturbation; c_0 the unperturbed sound speed. Indices 1 and 2 refer respectively to the fast and slow magnetosonic waves, index 3 to the Alfvén waves. The plus sign corresponds to a wave propagating in the direction of positive x, the minus sign to the direction of negative x. In addition,

$$u_{1,2} = \frac{1}{2}\left\{ \sqrt{c_0^2 + \frac{H_0^2}{4\pi\rho_0} + \frac{H_{0x}c_0}{\sqrt{4\pi\rho_0}}} \pm \sqrt{c_0^2 + \frac{H_0^2}{4\pi\rho_0} - \frac{H_{0x}c_0}{\sqrt{\pi\rho_0}}} \right\}, \qquad u_3 = \frac{H_{0x}}{4\pi\rho_0}.$$

We shall consider only boundary conditions relating the invariants $\Lambda_j^{\pm}$, of the form

$$\Lambda_i^+ - \mu\Lambda_i^- = 0 \quad (x = 0); \qquad \Lambda_i^+ - \mu\Lambda_i^- = \varepsilon\cos\omega t \quad (x = l);$$

$$\sum_k \left(\alpha_{jk}\Lambda_k^- + \beta_{jk}\Lambda_k^-\right) = 0 \quad (x = l,\ x = 0; \quad k \neq i,\ j \neq i),$$

which correspond to the excitation of only one type of wave ($\mu = \text{const}$). Conditions of this kind correspond, for example, to the prescription of velocities

$$v_x = v_y = v_z = 0 \quad (x = l);$$

$$v_x = v_z = 0, \qquad v_y = \varepsilon\cos\omega t \quad (x = 0).$$

In this paper only the excitation of fast and slow magnetosonic waves is considered.

In the linear case we shall have:

$$\Lambda_i^+ = -\frac{\varepsilon}{2\sin\omega l/u_i}\sin\left(\omega t - \frac{\omega}{u_i}x\right), \qquad \Lambda_i^- = -\frac{\varepsilon}{2\mu\sin\omega l/u_i}\sin\left(\omega t + \frac{\omega}{u_i}x\right),$$

$$\Lambda_j^{\pm} = 0 \quad (j \neq i). \tag{1}$$

On approaching resonance, $\omega/u_i \to \pi n/l$, the amplitude of the excited oscillations becomes much greater than the amplitude of the forcing force. To investigate and find the solution near resonance we must take into account corrections of the next order, assuming $|\Lambda_i| \gg \varepsilon$, $|\Lambda_i| \gg |\Lambda_j|$. In doing so we shall assume that the amplitude of the oscillations that arise remains small, i.e. $|\Lambda_i| \ll |\mathbf{H}_0|$.

Although the equations of magnetohydrodynamics do not have exact Riemann invariants (constant on the corresponding characteristics, as is the case for $\Lambda_j^{\pm}$ in the linear approximation), in the present case, restricting ourselves in the exact equations of magnetohydrodynamics to terms of first and second order in Λ_j inclusive, one can introduce approximate invariants $L^{\pm}(\Lambda_i^+, \Lambda_i^-)$ such that for them the equations will be approximately satisfied

$$\frac{\partial L^+}{\partial t} + u^+\frac{\partial L^+}{\partial x} = 0,$$

$$\frac{\partial L^-}{\partial t} + u^-\frac{\partial L^-}{\partial x} = 0 \tag{2}$$

Fig. 1

Figure 1: Fig. 1

and the boundary conditions

$$L^+ - \mu L^- + K(L^+, L^-) = 0, \qquad x = l;$$

$$L^+ - \mu L^- + K(L^+, L^-) = \varepsilon \cos \omega t, \qquad x = 0, \tag{3}$$

where K is a quantity of second order with respect to L; $u^\pm(L^+, L^-)$ are certain known functions of $L^\pm$.

Fig. 1

The remaining $\Lambda_j^\pm$ can be found by means of the known solution for $L^\pm$, and they are quantities of second order with respect to $L^\pm$ and are of lesser interest.

Equations (2) and the boundary conditions (3) make it possible to obtain the following approximate equation for determining

$$L^-(t) = L^-\big|_{x=0},$$

if terms of order higher than L^2 are neglected:

$$\mu \frac{dL^-}{dt} \left(\frac{2\Delta l}{u_i} + \frac{\alpha(1+\mu) l_0}{u_i^2} L^- \right) = \varepsilon \cos \omega t, \tag{4}$$

where Δl is the detuning; $2l/u_i = 2\pi/\omega + 2\Delta l/u_i$ and $l_0 = \pi u_i/\omega$; α is a certain constant coefficient depending on the unperturbed quantities.

Analysis of equation (4) shows that a continuous solution passing into (1) when

$$|\Delta l| \gg \left| \frac{\alpha(1+\mu) l_0}{u_i} L^- \right|$$

exists only up to a certain critical value of $|\Delta l|$ (it is assumed that $\varepsilon > 0,\ \mu > 0$)

$$\left| \frac{\Delta l}{l_0} \right| \gg \frac{1}{\pi} \sqrt{\frac{(1+\mu)\varepsilon\alpha}{\pi \mu u_i}},$$

while for smaller $|\Delta l|$ there is only a regime with a shock wave. One can find the completely corresponding solution, as well as the magnitude and position of the jump, from the requirement that the solution be real and that shock waves be impossible

rarefaction in magnetic hydrodynamics ([6]). The graph of the quantity $L^{-}(t)$ for $\mu > 0$, $\Delta l = 0$ and fast magnetosonic waves is shown in Fig. 1; the flow pattern is shown in Fig. 2, where the solid lines are the paths of shock waves. This solution is analogous to the corresponding solution in ordinary gas dynamics ([4]).

For the mean rate of irreversible heating in a shock wave at the maximum value of the discontinuity ($\Delta l = 0$), for a fast magnetosonic wave in a strong magnetic field we obtain

$$\overline{\frac{dE}{dt}} = \frac{\varepsilon H_0^2(1+\mu+\mu^2)}{6\pi^3\mu\sqrt{\rho_0}}\sqrt{\frac{\varepsilon}{6\mu(1+\mu)H_{0y}}}.$$

The formulas given lose their meaning at $\mu = -1$, which corresponds to boundary conditions on the free surface of a liquid held by an external magnetic field. In this case, in order to find the solution near resonance it is necessary to take into account terms of third order in L.

Consider the case when the magnetic field is parallel to the boundary of the liquid ($x = 0$ and $x = l$). In this case the equations of magnetic hydrodynamics reduce to the equations of gas dynamics with the effective pressure ([5])

$$p = p(\rho) + \frac{H_0^2\rho^2}{8\pi\rho_0^2}$$

and exact Riemann invariants can be introduced,

$$\Lambda^{\pm} = 2(c - c_0) \pm v_x.$$

Fig. 2

The boundary conditions in this case have the form $\Lambda^+ = -\Lambda^-$, $x = l$; $\Lambda^+ + \Lambda^- = \varepsilon\cos\omega t$, $x = 0$. For the quantity $\Lambda^-(t)$ one can obtain an equation analogous to (4):

$$\frac{d\Lambda^-}{dt}\left[\frac{2\Delta l}{c_0} + \frac{9}{8}\frac{l_0}{c_0^3}(\Lambda^-)^2 + \frac{1}{8c_0^3}\int_0^l(\Lambda^-)^2\,dx\right] = \varepsilon\cos\omega t. \tag{5}$$

A general analysis of this equation is difficult. It is easy to see that for $\Delta l = 0$ shock waves do not arise. However, it can be shown that for

$$\frac{\Delta l}{l_0} = -\frac{\pi\sqrt[3]{2}}{3\sqrt{3}\{\Gamma(4/3)\}^2}\left(\frac{\varepsilon}{3\pi c_0}\right)^{2/3},$$

where $\Gamma(4/3)$ is Euler' s gamma function, $\varepsilon > 0$, there is a jump (apparently, this case corresponds to its maximum value).

In this case the solution of (5) has the form

$$\Lambda^- = -\left(\frac{8\varepsilon c_0^2}{3\pi}\right)^{1/3}\left(\sin\frac{\pi c_0 t}{l_0}\right)^{1/3},$$

Fig. 3

and the jump arises near the points $t = 2\pi n/\omega$, in the neighborhood of which the written solution becomes inapplicable. In this case the magnitude of the discontinuity is of order $\varepsilon^{1/2}$, whereas Λ is a quantity of order $\varepsilon^{1/3}$, i.e., the discontinuity has a higher order of smallness than Λ itself. The flow pattern in the tx plane is presented in Fig. 3, where shock waves are shown by solid lines, and charac-

characteristics that bound the zone of the centered rarefaction wave. The mean rate of irreversible heating of the liquid in shock waves for a strong magnetic field is

$$\overline{\frac{dE}{dt}} = \frac{H_0^3}{5\sqrt[4]{\pi}}\left(\frac{\varepsilon\sqrt{\rho_0}}{H_0}\right)^{3/2}.$$

The case of excitation of Alfvén waves also requires consideration of terms of third order in $\Lambda_i^{\pm}$. In this case, however, it is necessary to take into account the excitation not of Alfvén invariants, which greatly complicates the problem.

The results obtained show that, for heating a liquid by magnetosonic waves, the creation of conditions for the excitation of natural oscillations with multiple frequencies is apparently the most advantageous, since the rate of irreversible heating does not depend on the magnitude of the dissipative coefficients and is proportional to the amplitude of the external field to the power 3/2.

Let us note that in a strongly magnetized plasma the equations of magnetohydrodynamics are approximately valid for transverse motion even in the absence of collisions ([7]). Although the exact mechanism of dissipation under such conditions is unknown, the formulas obtained make it possible to calculate the magnitude of the rate of dissipation, using only the supposition that shock waves exist.

Mathematical Institute named after V. A. Steklov
Academy of Sciences of the USSR

Received
9 V 1962

REFERENCES

[1] D. A. Frank-Kamenetskii, ZhTF, **30**, issue 8, 899 (1960). [2] D. A. Frank-Kamenetskii, ZhETF, **39**, issue 3, 669 (1960). [3] E. K. Zavoiskii, I. A. Kovan et al., ZhTF, **31**, issue 6, 513 (1961). [4] S. Veshov, Phys. Fluids, **1**, No. 3, 205 (1958). [5] L. D. Landau, E. M. Lifshitz, *Electrodynamics of Continuous Media*, Moscow, 1957. [6] S. V. Iordanskii, DAN, **121**, No. 4, 612 (1958). [7] G. Chew et al., Proc. Roy. Soc., A **236**, 112 (1956).

Note: Figure translations are in progress. See original paper for figures.