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Abstract

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EINSTEIN SPACES OF MAXIMAL MOBILITY

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1. For an island distribution of matter in general relativity, at infinity the geometry of space-time coincides with the geometry of special relativity and is characterized uniquely by the presence of a 10-term group of motions ((1), pp. 10, 244). In the case where the physical formulation of the problem admits an "internal" problem (the energy-momentum tensor $T_{ik} \neq 0$) and an "external" problem ($T_{ik} = 0$), for the latter the Einstein field equations will have the form $R_{ik} = 0$ or $R_{ik} = \varkappa g_{ik}$, depending on whether the "cosmological term" is zero or not. However, this term in the Einstein equations can always be assigned to T_{ik} ((1), p. 256). The study of many known solutions of the field equations $R_{ik} = \varkappa g_{ik}$ (for example, the Schwarzschild, Kottler, Weyl, and other metrics) shows that, as a rule, they are all characterized by more or less high mobility, and the group of motions of such spaces is either a subgroup of the group of motions of spaces of maximal mobility or coincides with it. On the other hand, Einstein spaces, in the sense of the algebraic structure of the curvature tensor, are divided into three types ($T^*_k, k = 1, 2, 3$), for each of which one can indicate, in a nonholonomic orthonormal frame, a canonical form of the components of the curvature tensor ((2), § 19); as a consequence there arises the question of spaces of maximal mobility for each of these three types. The case of free spaces ($\varkappa = 0$) was considered by A. Z. Petrov ((2), Ch. IV). In the present note the question of maximally mobile Einstein spaces ($n = 4$, signature of hyperbolic type)

$$R_{ik} = \varkappa g_{ik}, \qquad \varkappa = \text{const} \neq 0, \tag{1}$$

is investigated, and the metrics of such spaces are indicated. In conclusion, a complete summary of maximally mobile Einstein spaces is given, including also the case of free spaces considered in (2).

2. We shall state a known theorem ((3), § 53), to which we shall refer below.

Theorem 1. *In order that V_n admit a group of motions G_r, it is necessary and sufficient that the rank of the system of equations*

$$LR_{ijkl} = 0, \qquad LR_{ijkl,m_1} = 0, \dots, \qquad LR_{ijkl,m_1 \dots m_q} = 0 \tag{2}$$

(L denotes the Lie derivative in the direction of the vector ξ^k) up to the number q with respect to the unknowns ξ_k and $\xi_{k,l}$ (under the condition $\xi_{(k,l)} = 0$) be equal to

$$p = \frac{n(n+1)}{2} - r,$$

and the addition of the equations from (2) with number $q+1$ should not change the rank.

Let us consider spaces T_1^*. Comparing at a given point of space-time the representation of its curvature tensor in a three-dimensional com-

complex space:

$$R_{ijkl} \to R_{ab}, \qquad a \to [ij],\ b \to [kl]; \qquad \Pi_{AB} = R_{ab} + iR_{ab+3}, \tag{3}$$

we obtain for the quantities Π_{AB} the expression $((2), \S 19)$:

$$\Pi_{AB} = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix}, \qquad k_1 + k_2 + k_3 = -\varkappa, \tag{4}$$

and the matrix of coefficients of ξ^k and $\xi_{k,l}$ of the first relation of system (2) will take the form:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \Pi_{11,\sigma} \\ 0 & 0 & i(k_1 - k_2) & 0 & 0 & (k_2 - k_1) & \Pi_{12,\sigma} \\ 0 & i(k_3 - k_1) & 0 & 0 & (k_1 - k_3) & 0 & \Pi_{13,\sigma} \\ 0 & 0 & 0 & 0 & 0 & 0 & \Pi_{22,\sigma} \\ i(k_2 - k_3) & 0 & 0 & (k_3 - k_2) & 0 & 0 & \Pi_{23,\sigma} \\ 0 & 0 & 0 & 0 & 0 & 0 & \Pi_{33,\sigma} \end{pmatrix}.$$

Assuming that T_1^* admits G_r $(r \geq 7)$, on the basis of Theorem 1 we set all second-order determinants equal to zero. We have

$$k_s = -\frac{\varkappa}{3} \quad (s = 1, 2, 3). \tag{5}$$

Conditions (5) are necessary and sufficient for T_1^* to be a space of constant curvature $((2), \S 16)$. If, however, one considers the case when spaces of the first type admit transitive groups of motions G_r for $5 \leq r < 7$, then by the same method we obtain that the desired spaces must be symmetric and reducible $((2), \S 47)$. Their metric in a special coordinate system is written as follows:

$$
\begin{aligned}
&1) \quad ds^2 = -\operatorname{ch}^2(\sqrt{\varkappa}\, x^2)\, dx^{1^2} - dx^{2^2} - \cos^2(\sqrt{\varkappa}\, x^4)\, dx^{3^2} + dx^{4^2}, \qquad \varkappa > 0;\\
&2) \quad ds^2 = -\cos^2(\sqrt{-\varkappa}\, x^2)\, dx^{1^2} - dx^{2^2} - \operatorname{ch}^2(\sqrt{-\varkappa}\, x^4)\, dx^{3^2} + dx^{4^2}, \qquad \varkappa < 0;
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
&3) \quad ds^2 = -\operatorname{ch}^2(\sqrt{\varkappa}\, x^2)\, dx^{1^2} - dx^{2^2} + \operatorname{ch}^2(\sqrt{\varkappa}\, x^4)\, dx^{3^2} - dx^{4^2}, \qquad \varkappa > 0;\\
&4) \quad ds^2 = -\cos^2(\sqrt{-\varkappa}\, x^2)\, dx^{1^2} - dx^{2^2} + \cos^2(\sqrt{-\varkappa} x^4)\, dx^{3^2} - dx^{4^2}, \qquad \varkappa < 0.
\end{aligned}
$$

3. Consider the spaces T_2^*. The curvature tensor in the complex orthoframe is reduced to the form

$$
\Pi_{AB} = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 + 1 & i \\ 0 & i & k_2 - 1 \end{pmatrix}, \qquad k_1 + 2k_2 = -\varkappa. \tag{7}
$$

Starting from (7), it is easy to obtain the conditions imposed on the curvature tensor:

$$
1) \quad k_1 = k_2: \quad l^i \left(R_{ijkl} - \frac{\varkappa}{3} g_{ijkl} \right) = l^i \varepsilon_{ijpq} \left(R^{pq}_{kl} - \frac{\varkappa}{3} g^{pq}_{kl} \right) = 0, \quad L^{lk} = \omega^{lk}_s; \tag{8}
$$

$$
2) \quad k_1 \neq k_2: \quad l^{il^{k(R}}_{ijkl} + \alpha_1 g_{ijkl}) = l^{il^{k(\varepsilon}}_{ijpq} R^{pq}_{kl} + \beta_1 g_{ijkl}) = 0, \quad l^{kl}_{h,k} = a l_h,
$$

where $l^k l_k = 0$; ε_{ijpq} is the discriminant tensor; $\alpha_1 + i\beta_1 = k_1$. Then, using Theorem 1 with respect to the matrix of system (2) for spaces of the second type and conditions (8), one can show that the following two theorems are valid:

Theorem 2. *The maximally mobile spaces T_2 ($\varkappa = 0$) are two classes of spaces with metrics:*

1)

$$
ds^2 = 2dx^1 dx^4 + \varepsilon x^4 \left(\operatorname{ch}^2(\lambda \ln x^4)\, dx^{2^2} + \cos^2(\mu \ln x^4)\, dx^{3^2} \right), \tag{9}
$$

where

$$
\lambda = \frac{\sqrt{a^2 + 4}}{2a}, \qquad \mu = \frac{\sqrt{4 - a^2}}{2a}, \qquad a^2 < 4, \qquad \varepsilon = \pm 1, \qquad a = \text{const};
$$

2)

$$
ds^2 = 2dx^1 dx^4 + A(x^4)\, dx^{2^2} + 2B(x^4)\, dx^2 dx^3 + C(x^4)\, dx^{3^2}, \tag{9'}
$$

where

$A = -\Delta\theta'$, $B = \Delta\psi'$, $C = -\Delta\varphi'$, $\Delta = AC - B^2$, $\varphi' = 2b\varphi\psi - \psi^2 + \varphi^2 - 1$,

$\psi' = \psi(\varphi - \theta) + b(1 + \psi^2 + \theta\varphi)$, $\theta' = 2b\psi\theta + \psi^2 - \theta^2 - 1$, $b^2 = \Delta(\psi + b\theta)^2$, $b = \text{const}$.

They admit a six-dimensional group of motions.

Theorem 3. *The maximal group of motions for T_2^* ($\varkappa \neq 0$) is a group of the fifth order. A space of maximal mobility is possible only for $\varkappa > 0$ and in a semigeodesic coordinate system has the form*

$$ds^2 = -e^{2\lambda x^4}\left(2dx^1dx^3 + dx^{2^2}\right) + \varepsilon e^{-\lambda x^4}dx^{3^2} - dx^{4^2},$$

$$\lambda = \sqrt{\frac{\varkappa}{3}}, \qquad \varepsilon = \pm 1. \tag{9"}$$

4. For spaces of the third type T_3^* ($\varkappa \neq 0$)

$$\Pi_{AB} = \begin{pmatrix} -\frac{1}{3}\varkappa & -1 & 0 \\ -1 & -\frac{1}{3}\varkappa & -i \\ 0 & -i & -\frac{1}{3}\varkappa \end{pmatrix}$$

we obtain properties analogous to properties (8):

$$l^i l^k \left(R_{ijkl} - \frac{\varkappa}{3}g_{ijkl}\right) = l^i l^k \varepsilon_{ijpq} R^{pq}_{kl} = 0, \qquad Ll^k = \omega l^k, \tag{10}$$

$$n^i n^k \left(R_{ijkl} - \frac{\varkappa}{3}g_{ijkl}\right) = n^i l^k \varepsilon_{ijpq}\left(R^{pq}_{kl} - \frac{\varkappa}{3}g^{pq}_{kl}\right) = 0, \qquad Ln^k = \omega n^k,$$

where $l^k l_k = l^k n_k = 0$, ε_{ijpq} is the discriminant tensor.

Conditions (10) and the relations obtained from the matrix of system (2) for the third type make it possible to determine completely the group and the metric of a space of maximal mobility:

Theorem 4. *A maximally mobile space T_3^* ($\varkappa \neq 0$) is possible only for $\varkappa > 0$. It admits a transitive G_4 and has a metric of the form*

$$ds^2 = e^{-2\lambda x^4}\left(\pm 2dx^1dx^3 - dx^{2^2}\right) \pm 2e^{\lambda x^4}dx^2dx^3 - \frac{1}{2}e^{4\lambda x^4}dx^{3^2} - dx^{4^2}, \qquad \lambda = \sqrt{\varkappa/3}. \tag{11}$$

5. Combining the above conclusions with the results obtained in (^{2}), Ch. IV, for spaces $R_{ik} = 0$, we have the following summary of maximally mobile Einstein spaces ($R_{ik} = \varkappa g_{ik}$, $\varkappa = \text{const}$, $n = 4$, signature $(---+)$):

Type I: $\varkappa = 0$—Minkowski space;
$\varkappa \neq 0$—space of constant curvature.

Type II: $\chi = 0$—a six-dimensional group, metrics (9) and (9′);
$\chi \neq 0$—a five-dimensional group, metric (9″).

Type III: $\chi = 0$—

$$ds^2 = \pm\left(x^2\, dx^{1^2} + dx^{2^2}\right) + 2dx^3dx^4 + \left(\left(x^3 \pm \frac{1}{4}x^{2^2}\right)\ln(ax^2) \mp x^{2^2} + b\right) dx^{4^2},$$

$(a, b = \text{const})$—the space admits a non-Abelian G_2;

$\chi \neq 0$—a group transitive G_4, metric (11).

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Note: Figure translations are in progress. See original paper for figures.

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