

# AFFINE CONNECTIONS ON MANIFOLDS WITH AN ALMOST COMPLEX STRUCTURE

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**Abstract**

**Full Text**

**MATHEMATICS**

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## **AFFINE CONNECTIONS ON MANIFOLDS WITH AN ALMOST COMPLEX STRUCTURE**

*(Presented by Academician P. S. Aleksandrov, October 5, 1961)*

Let  $M_{2n}$  be a  $2n$ -dimensional manifold with an almost complex structure, i.e., on  $M_{2n}$  there is given a tensor  $F_j^i$  ( $i, j, k, l = 1, \dots, 2n$ ) such that  $F_j^i F_k^j = -\delta_k^i$  (1). Let  $\Gamma_{jk}^i$  be a symmetric affine connection on  $M_{2n}$ . The expression  $\tilde{\Gamma}_{jk}^i = \Gamma_{jk}^i + F_j^i F_{l,k}^l$  (the comma denotes covariant differentiation with respect to  $\Gamma_{jk}^i$ ) defines on  $M_{2n}$  an affine connection whose parallel transport from a point  $x_1$  to a point  $x_2$  along a curve  $L$  is carried out as follows: the vector is first subjected to the automorphism  $F(x_1) = (F_j^i(x_1))$ , then transported from  $x_1$  to  $x_2$  along  $L$  parallel with respect to the connection  $\Gamma_{jk}^i$ , and, finally, subjected to the inverse automorphism  $F^{-1}(x_2) = -F(x_2)$ . We shall call the connection  $\tilde{\Gamma}_{jk}^i$  the conjugate connection. The connection  $\tilde{\Gamma}_{jk}^i$ , generally speaking, is not symmetric. We also introduce the connection  $\gamma_{jk}^i = \frac{1}{2}(\tilde{\Gamma}_{jk}^i + \tilde{\Gamma}_{kj}^i)$ , which we shall call the symmetric conjugate connection. Notice that if the connection  $\Gamma_{jk}^i$  is almost complex (1), then  $\Gamma_{jk}^i = \tilde{\Gamma}_{jk}^i = \gamma_{jk}^i$ , and conversely.

Consider the tensors:

$$A_{jk}^i = -(F_j^l F_{l,k}^i + F_k^l F_{l,j}^i);$$

$$B_{jk}^i = F_j^l (F_{k,l}^i - F_{l,k}^i);$$

$$B_i = B_{il}^l = B_{li}^l = F_m^l F_{i,l}^m;$$

$$P_{jk}^i = B_{jk}^i - \frac{1}{2n} \delta_j^i B_k - \frac{1}{2n} \delta_k^i B_j - \frac{1}{2n} F_j^i F_k^l B_l - \frac{1}{2n} F_k^i F_j^l B_l.$$

Notice that  $A_{li}^l = A_{il}^l = 0$ ;  $P_{li}^l = P_{il}^l = 0$ .

The following formulas hold:

$$A_{jk}^i - A_{kj}^i = B_{jk}^i - B_{kj}^i = P_{jk}^i - P_{kj}^i = 4t_{jk}^i,$$

where

$$t_{kj}^i = \frac{1}{4} \left[ F_j^l \left( \frac{\partial F_k^i}{\partial x^l} - \frac{\partial F_l^i}{\partial x^k} \right) - F_k^l \left( \frac{\partial F_j^i}{\partial x^l} - \frac{\partial F_l^i}{\partial x^j} \right) \right]$$

is the torsion tensor of the almost complex structure (1).

The geometric meaning of the tensors  $A_{jk}^i$ ,  $B_{jk}^i$  is clarified by

**Theorem 1.**

$$B_{jk}^i = -F_l^i F_j^m S_{mk}^l; \quad A_{jk}^i = 2(\Gamma_{jk}^i - \gamma_{jk}^i) - B_{kj}^i,$$

where  $S_{jk}^i = \hat{\Gamma}_{jk}^i - \tilde{\Gamma}_{kj}^i$  is the torsion tensor of the conjugate connection.

The following theorem clarifies the geometric meaning of the torsion tensor of the almost complex structure.

**Theorem 2.** Let  $\Gamma_{jk}^i$  be an arbitrary symmetric affine connection;  $\gamma_{jk}^i$  its symmetric conjugate connection;  $\tilde{\gamma}_{jk}^i$  the connection conjugate to the connection  $\gamma_{jk}^i$ . Then the torsion tensor of the connection  $\tilde{\gamma}_{jk}^i$  does not depend on the choice of the initial connection  $\Gamma_{jk}^i$  and is equal to twice the torsion tensor of the almost complex structure.

The quantities

$$\Pi_{jk}^i = \Gamma_{jk}^i - \frac{1}{2n+1} \delta_j^i \Gamma_{lk}^l - \frac{1}{2n+1} \delta_k^i \Gamma_{lj}^l$$

are called the Thomas projective parameters. The projective parameters form a geometric object whose specification is equivalent to specifying a family of geodesic lines of some pencil of geodesically equivalent spaces of affine connection (2).

**Definition.** A symmetric affine connection for which  $B_i = 0$  will be called a normal affine connection.

**Theorem 3.** For any symmetric affine connection there exists, and moreover uniquely, a normal affine connection geodesically equivalent to it. The components of this connection have the form

$$\hat{\Gamma}_{jk}^i = \Gamma_{jk}^i - \frac{1}{2n} \delta_j^i F_m^l F_{k,l}^m - \frac{1}{2n} \delta_k^i F_m^l F_{j,l}^m.$$

**Corollary.** The projective parameters  $\Pi_{jk}^i$  invariantly determine the unique normal affine connection with the same geodesic lines. Its components have the form

$$\hat{\Gamma}_{jk}^i = \Pi_{jk}^i - \frac{1}{2n} \delta_j^i \left( F_m^l F_k^r \Pi_{lr}^m + F_m^l \frac{\partial F_k^m}{\partial x^l} \right) - \frac{1}{2n} \delta_k^i \left( F_m^l F_j^r \Pi_{lr}^m + F_m^l \frac{\partial F_j^m}{\partial x^l} \right).$$

**Theorem 4.** If two symmetric affine connections are geodesically equivalent, then the tensors  $A_{jk}^i$  and  $P_{jk}^i$  of these connections coincide.

The proof follows from the relations

$$\begin{aligned}
 A_{jk}^i &= \Pi_{jk}^i - F_j^l F_k^m \Pi_{lm}^i + F_l^i F_j^m \Pi_{mk}^l + F_l^i F_k^m \Pi_{jm}^l - F_j^l \frac{\partial F_l^i}{\partial x^k} - F_k^l \frac{\partial F_j^i}{\partial x^l}, \\
 P_{jk}^i &= \Pi_{jk}^i + F_j^l F_k^m \Pi_{lm}^i - \frac{1}{2n} \delta_j^i F_l^m F_k^r \Pi_{mr}^l - \frac{1}{2n} \delta_k^i F_l^m F_j^r \Pi_{mr}^l + \\
 &+ \frac{1}{2n} F_j^i F_l^m \Pi_{mk}^l + \frac{1}{2n} F_k^i F_l^m \Pi_{jl}^m + \frac{1}{2n} \delta_j^i F_k^l \frac{\partial F_l^m}{\partial x^m} + \frac{1}{2n} \delta_k^i F_j^l \frac{\partial F_l^m}{\partial x^m} - \\
 &- \frac{1}{2n} F_j^i \frac{\partial F_k^l}{\partial x^l} - \frac{1}{2n} F_k^i \frac{\partial F_j^l}{\partial x^l} - F_j^l \frac{\partial F_l^i}{\partial x^k} + F_j^l \frac{\partial F_k^i}{\partial x^l}.
 \end{aligned}$$

**Definition.** Let  $\Pi_{jk}^i$  be the projective parameters;  $\Gamma_{jk}^i$  the associated normal affine connection;  $\tilde{\Gamma}_{jk}^i$  and  $\gamma_{jk}^i$ , respectively, the conjugate and symmetric conjugate connections;  $\tilde{\Pi}_{jk}^i$  and  $\pi_{jk}^i$  the projective parameters corresponding to the connections  $\tilde{\Gamma}_{jk}^i$  and  $\gamma_{jk}^i$ . The parameters  $\tilde{\Pi}_{jk}^i$  and  $\pi_{jk}^i$  will be called, respectively, conjugate and symmetric conjugate projective parameters.

The following two propositions are analogous to Theorems 1 and 2.

**Theorem 5.**

$$P_{jk}^i = -F_l^i F_j^m R_{mk}^l; \quad A_{jk}^i = 2(\Pi_{jk}^i - \pi_{jk}^i) - P_{kj}^i,$$

where

$$R_{jk}^i = \tilde{\Pi}_{jk}^i - \Pi_{jk}^i$$

is a tensor that it is natural to call the torsion tensor of the conjugate projective parameters.

**Theorem 6.** Let  $\Pi_{jk}^i$  be arbitrary symmetric projective parameters; let  $\pi_{jk}^i$  be their symmetric conjugate parameters; and let  $\tilde{\pi}_{jk}^i$  be the conjugate parameters for  $\pi_{jk}^i$ . Then the torsion tensor of the parameters  $\tilde{\pi}_{jk}^i$  does not depend on the choice of the initial parameters  $\Pi_{jk}^i$  and is equal to twice the torsion tensor of the almost complex structure.

**Theorem 7.** Let  $\Pi_{jk}^i$  be symmetric projective parameters, and let  $P_{jk}^i$  be the tensor obtained from  $\Pi_{jk}^i$  by the formula of Theorem 4. The normal affine connection associated with the projective parameters  $\Pi_{jk}^i$  is the unique connection of the associated bundle of geodesically equivalent affine connections such that the torsion tensor of the projective parameters corresponding to its conjugate connection is equal to  $-F_l^i F_j^m P_{mk}^l$ .

The following theorem clarifies the geometric meaning of normality of an affine connection.

**Theorem 8.** In order that a symmetric affine connection be normal, it is necessary and sufficient that, for every  $2n$ -covariant skew-symmetric tensor  $\xi_{i_1 \dots i_{2n}}$  (volume element) and for every curve  $L$ , parallel transport of this tensor along  $L$  in the connection  $\Gamma_{jk}^i$  and in the symmetric conjugate connection  $\gamma_{jk}^i$  give one and the same result.

It is well known <sup>(1)</sup> that a symmetric almost complex connection can exist only in the case when the almost complex structure is pseudocomplex, i.e., when the torsion tensor of the structure is equal to zero.

**Theorem 9.** In order that a symmetric affine connection on a manifold with a pseudocomplex structure be almost complex, it is necessary and sufficient that  $A_{jk}^i = B_{jk}^i = 0$ . In order that this connection be geodesically equivalent to an almost complex connection, it is necessary and sufficient that  $A_{jk}^i = P_{jk}^i = 0$ .

**Theorem 10.** The connections  $\hat{\Gamma}_{jk}^i = \frac{1}{2}(\Gamma_{jk}^i + \tilde{\Gamma}_{jk}^i)$  and  $\tilde{\pi}_{jk}^i = \frac{1}{2}(\Gamma_{jk}^i + \gamma_{jk}^i) - \frac{1}{4}B_{jk}^i$  are almost complex. The torsion form of the connection  $\tilde{\Gamma}_{jk}^i$  coincides with the torsion form of the almost complex structure.

The connections of Theorem 10 have been considered earlier. Lichnerowicz calls the first of them the almost complex connection induced by the connection  $\Gamma_{jk}^i$ ; he constructs the second connection in order to prove the existence of an almost complex connection whose torsion form is equal to the torsion form of the structure <sup>(1)</sup>.

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