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Abstract

Full Text

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ON THE EXISTENCE OF A LINEAR POLYNOMIAL OPERATOR GIVING THE BEST APPROXIMATION ON A CLASS OF FUNCTIONS

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1. Let $C_{2\pi}$ be the space of continuous functions of period 2π with norm $\|f\| = \max_x |f(x)|$, and let the operator U_n, for fixed $n = 0, 1, 2, \ldots$, assign to each function $f \in C_{2\pi}$ a trigonometric polynomial $U_n(f, x)$ of degree not exceeding n. Let $\mathfrak{M}$ be some set of functions in the space $C_{2\pi}$. We consider the question of the existence of a linear operator U_n such that

$$\sup_{f\in\mathfrak{M}} \|U_n(f,x) - f(x)\| = \sup_{f\in\mathfrak{M}} E_n(f), \tag{1}$$

where $E_n(f)$ is the best uniform approximation of the function $f \in C_{2\pi}$ by trigonometric polynomials of degree not exceeding n.

It is known [(1–4)] that if $\mathfrak{M}$ is the class $W^{(r)}M$ $(r > 0)$ of functions f for which $|f^{(r)}(x)| \le M$, then for any $n = 0, 1, \ldots$ there exists a linear polynomial operator U_n for which (1) holds; that is, on this class there exists a best linear method of approximation. For integral r such a method is furnished, with a suitable choice of the multipliers $\lambda_k^{(n)}$, by the sequence of polynomials

$$U_n(f,x) = U_n(f,x,\lambda) = \frac{a_0}{2} + \sum_{k=1}^{n} \lambda_k^{(n)} (a_k \cos kx + b_k \sin kx), \tag{2}$$

$$n = 0, 1, 2, \ldots,$$

where a_0, a_k, b_k are the Fourier coefficients of the function f.

At the same time [(5)], on the class H_ω of functions of the space $C_{2\pi}$ whose modulus of continuity $\omega(f,t)$ does not exceed a given concave upward modulus of continuity $\omega(t)$, the polynomials (2), for no choice of $\lambda_k^{(n)}$, can realize equality (1), provided only that $\omega(t)$ is nonlinear on the interval $\left[0, \frac{\pi}{n+1}\right]$. In particular, this will be the case if $\omega(t) = Kt^\alpha$ $(0 < \alpha < 1,\ 0 \le t \le \pi)$.

We prove here that, for the class H_ω under the stated assumptions on $\omega(t)$, the answer to the question posed remains negative, whatever the linear operators U_n may be.

Let us first note that if some operator U_n realizes equality (1) on the set $\mathfrak{M}$, then, for any function $f_0 \in \mathfrak{M}$ such that

$$E_n(f_0) = \sup_{f \in \mathfrak{M}} E_n(f),$$

the polynomial $U_n(f_0, x)$ must identically coincide with the polynomial of best approximation of the function f_0 (by virtue of the uniqueness of the latter).

Let f_0 be an arbitrary function from H_ω ($\omega(t)$ concave upward), and let $x_1, x_2, \dots, x_{2n+2}$ be points of the interval $[0, 2\pi]$ at which the difference between $f_0(x)$ and its polynomial of best approximation $T_n(f_0, x)$ takes alternately the values $\pm E_n(f_0)$. Denote by θ the largest of the intervals (x_k, x_{k+1}) $(k = 1, 2, \dots, 2n+2;\ x_{2n+3} = x_1 + 2\pi)$; let it be, for-

for example, (x_m, x_{m+1}), and by δ its length. The continuous function $f_1(x)$, coinciding with $f_0(x)$ outside θ and linear on θ, belongs to the class H_ω. Let us next consider the function

$$\psi(x) = \begin{cases} f_0(x), & (x \notin \theta), \\ \inf\limits_{t \in \theta} \left[f_0(t) + \omega\left(|x - t|\right) \right], & (x \in \theta). \end{cases}$$

It is not hard to verify that $\psi \in H_\omega$, $\psi(x)$ is convex upward on the interval θ, and, if $\omega(t)$ is nonlinear on the segment $[0, \delta]$, then $\psi(x) - f_1(x) > 0$ for all $x_m < x < x_{m+1}$.

Denote by x_0 the midpoint of the interval (x_m, x_{m+1}); note further that the function $g(x)$, equal to $f_0(x)$ outside θ, taking at the point x_0 the value $\psi(x_0)$, and linear on the intervals $[x_m, x_0]$ and $[x_0, x_{m+1}]$, also belongs to H_ω.

If $E_n(f_0) = \sup\limits_{f \in H_\omega} E_n(f)$, then $E_n(g) = E_n(f_1) = E_n(f_0)$, and for the functions f_0, f_1, and g the polynomial of best approximation is one and the same polynomial $T_n(f_0, x)$.

Suppose now that for the operator $\overline{U}_n$, for some fixed n, the equality

$$\sup_{f \in H_\omega} \left\| \overline{U}_n(f, x) - f(x) \right\| = \sup_{f \in H_\omega} E_n(f) \tag{3}$$

is fulfilled. Then, in view of the remark made above,

$$\overline{U}_n(g, x) \equiv \overline{U}_n(f_1, x) \equiv T_n(f_0, x). \tag{4}$$

Let us show that the operator $\overline{U}_n$ cannot be linear. Indeed, let $g(x_0) - f_1(x_0) = \beta$, where $\beta > 0$. Put

$$\varphi(x) = \frac{1}{\beta}\,\omega\left(\frac{1}{2}\delta\right)\;[g(x) - f_1(x)].$$

It is clear that $\varphi \in H_\omega$, $\max_x |\varphi(x)| = \omega(\frac{1}{2}\delta)$, and, if we assume $\overline{U}_n$ linear, then, by (4), $\overline{U}_n(\varphi, x) \equiv 0$.

Consequently, if $\omega(t)$ is nonlinear on $\left[0, \dfrac{\pi}{n+1}\right]$, then, taking into account the exact value of the upper bound of the best approximations on the class H_ω ([5]), we obtain

$$\left\|\overline{U}_n(\varphi, x) - \varphi(x)\right\| = \omega\left(\frac{1}{2}\delta\right) \geq \omega\left(\frac{\pi}{2(n+1)}\right) > \frac{1}{2}\omega\left(\frac{\pi}{n+1}\right) = \sup_{f\in H_\omega} E_n(f),$$

in contradiction with (3).

We note that the proof of the nonexistence of a linear operator $\overline{U}_n$ realizing equality (3) can be carried out without relying on the relation

$$\sup_{f\in H_\omega} E_n(f) = \frac{1}{2}\omega\left(\frac{\pi}{n+1}\right).$$

One could, for example, show that the assumption of the linearity of $\overline{U}_n$ implies that $\overline{U}_n(f, x) \equiv 0$ for any $f \in H_\omega$, which also contradicts (3).

If $\omega(t)$ is convex upward and, for $0 \leq t \leq \gamma$ $(0 < \gamma < \pi)$, $\omega(t) = Kt$, then for sufficiently large n

$$\sup_{f\in H_\omega} E_n(f) = \sup_{f\in KH^{(1)}} E_n(f),$$

where $KH^{(1)} = W^{(1)}K$ is the class of functions $f \in C_{2\pi}$ satisfying on the whole axis the Lipschitz condition of degree 1 with constant K, and equality (3) will be realized by linear operators of the form (2).

Thus, the following assertion holds:

Theorem. *In order that, for fixed n, some linear polynomial operator $\overline{U}_n$ on the class H_ω, where $\omega(t)$ is convex upward, realize equality (3), it is necessary and sufficient that the function $\omega(t)$ be linear on the interval $\left[0, \dfrac{\pi}{n+1}\right]$.*

If there exists a finite limit $\lim_{t\to 0+} \frac{\omega(t)}{t} = K > 0$, then from the convexity of $\omega(t)$ it follows that $\omega(t) \leq Kt$ $(0 \leq t \leq \pi)$ and $H_\omega \subset KH^{(1)}$. Therefore, as is not difficult to see, there exists a sequence of linear operators $\{\bar{U}_n\}$ of the form (2) such that

$$\lim_{n\to\infty} \frac{\sup_{f\in H_\omega} \|\bar{U}_n(f,x,\lambda)-f(x)\|}{\sup_{f\in H_\omega} E_n(f)} = \lim_{n\to\infty} \frac{\sup_{f\in KH^{(1)}} E_n(f)}{\sup_{f\in H_\omega} E_n(f)} = 1, \tag{5}$$

i.e. in this case one may speak of the existence of an asymptotically best linear method of approximation.

2. Let us consider the analogous problem for one class of functions continuous on an interval. Let $C_{[-1,1]}$ be the space of functions continuous on the interval $[-1,1]$, with norm $\|f\| = \max_{-1\le x\le 1} |f(x)|$; $E_n(f;-1,1)$ is the best approximation of a function $f \in C_{[-1,1]}$ by algebraic polynomials of degree $\le n$ on this interval, and $P_n(f,x)$ is the polynomial of best approximation for f on $[-1,1]$.

The following assertion holds, which, incidentally, is also valid in the corresponding formulation for the periodic case.

Let $f \in C_{[-1,1]}$, and suppose that at the points $x_1, x_2, \dots, x_{n+2}$ of the interval $[-1,1]$ the difference $f(x) - P_n(f,x)$ alternately assumes the values $\pm E_n(f;-1,1)$. Then there will be two neighboring points, for example x_m and x_{m+1}, such that

$$|f(x_{m+1}) - f(x_m)| \ge 2E_n(f;-1,1). \tag{6}$$

Denote by $H^{(\alpha)}_{[-1,1]}$ the class of functions $f \in C_{[-1,1]}$ satisfying on the interval $[-1,1]$ the Lipschitz condition of order α $(0 < \alpha < 1)$ with constant 1. It is known[5] that as $n \to \infty$

$$\sup_{f\in H^{(\alpha)}_{[-1,1]}} E_n(f;-1,1) = \frac{\pi^\alpha}{2(n+1)^\alpha} + o\left(\frac{1}{n^\alpha}\right). \tag{7}$$

Suppose that for some n a function $f_0 \in H^{(\alpha)}_{[-1,1]}$ realizes the exact upper bound of the best approximations on this class. If n is sufficiently large, then, by (6) and (7), for the function f_0 there exists an interval (x_m, x_{m+1}) between neighboring alternation points with length differing arbitrarily little from

$$\frac{\pi}{n+1}.$$

Let the operator $\bar{V}_n$, mapping the space $C_{[-1,1]}$ onto the set of algebraic polynomials $\bar{V}_n(f,x)$ of degree not exceeding n, be such that

$$\sup_{f\in H^{(\alpha)}_{[-1,1]}} \|\bar{V}_n(f,x) - f(x)\| = \sup_{f\in H^{(\alpha)}_{[-1,1]}} E_n(f;-1,1) = E_n(f_0;-1,1). \tag{8}$$

If one assumes $\bar{V}_n$ to be linear, then for $0 < \alpha < 1$, arguing as in the periodic case, one can construct a function $\varphi \in H^{(\alpha)}_{[-1,1]}$ such that

$$\max_{-1\le x\le 1} |\varphi(x)| = \left(\frac{\pi}{2(n+1)}\right)^{\alpha} + o\left(\frac{1}{n^{\alpha}}\right) \quad \text{and} \quad \bar{V}_n(\varphi, x) = 0.$$

Taking (7) into account, we conclude that for sufficiently large n

$$\|\bar{V}_n(\varphi, x) - \varphi(x)\| > E_n(f_0; -1, 1),$$

i.e. the operator realizing equality (8) cannot be linear.

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Note: Figure translations are in progress. See original paper for figures.