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Abstract

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ON A DISCONTINUOUS POINT TRANSFORMATION OF THE LINE INTO THE LINE

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1. The investigation of a number of problems in the theory of oscillations and the theory of automatic control reduces to the study of a point transformation of the line into the line. A continuous point transformation of the line into the line has been studied sufficiently well; a discontinuous one, however, up to the present time, with the exception of the piecewise-linear case with one discontinuity of continuity ([9]), has not been studied in general form, and occurs only in particular problems of the theory of oscillations and the theory of automatic control ([1–8]).

In the present note we set forth the results of an investigation of a point transformation T of the line into the line of the form

$$\bar{x} = T(\mu)x = \begin{cases} T_1(x) = a(\mu) + f(\mu, x), & x < 0, \\ T_2(x) = b(\mu) + \varphi(\mu, x), & x > 0, \end{cases}$$

depending on the parameter μ ($\mu \geq 0$), under the assumptions that: 1) $a(0) = b(0) = 0$; 2) $f(\mu, 0) = \varphi(\mu, 0) = 0$; 3) $a(\mu)$ and $b(\mu)$ are continuously differentiable; 4) $f(\mu, x)$ is continuous and has first and second derivatives with respect to x, continuous in μ and x, for $x \leq 0$; 5) $\varphi(\mu, x)$ is continuous and has first and second derivatives with respect to x, continuous in μ and x, for $x \geq 0$. In other words, we consider a transformation continuous for $\mu = 0$, for which, when $\mu > 0$, one discontinuity of the first kind appears. A discontinuous piecewise-linear transformation of the line into the line of this type was considered in greater detail in the author' s papers ([9]).

As is known ([10]), if x^* is a fixed point of multiplicity n of the transformation T, i.e. $T^n x^* = x^*$, while $T^k x^* \neq x^*$ ($k < n$), then the points $Tx^*, T^2x^*, \dots, T^{n-1}x^*$ are also fixed points of T of multiplicity n; together with x^* they form the so-called n-member cycle of the transformation T; all fixed points of one cycle have equal characteristic roots.

2. It has been established that, in the cases considered, as a result of the appearance of a discontinuity of the transformation T for $\mu > 0$ at the point $x = 0$, cycles of stable fixed points of the transformation T of arbitrarily large multiplicity may appear, but of a quite definite type. Moreover, in a number of cases the transformation T has no stable fixed points, while on the x-axis there exists such a region G that $T^p x \in G$ for any point $x \in G$ and any natural number p *.

In order to characterize as briefly as possible the cycles of fixed points of the transformation T, let us denote by $T_{i_1 j_1}$ the product of transformations $T_1^{i_1} T_2^{j_1}$, by $T_{i_1 j_1 i_2 j_2}$ the product of transformations $T_{i_1 j_1}^{i_2} \times T_{i_1-\eta_1, j_1+1-\eta_1}^{j_2}$, ⋯, by $T_{i_1 j_1 \dots i_N j_N}$ the product of transformations $T_{i_1 j_1 \dots i_{N-1} j_{N-1}}^{i_N} T_{i_1 j_1 \dots i_{N-2} j_{N-2}, i_{N-1}-\eta_{N-1}, j_{N-1}+1-\eta_{N-1}}^{j_N}$, where i_s and j_s $(s = 1, 2, \dots, N)$ are natural, nonzero numbers, and i_s and j_s cannot, for the same s, both be greater than one simultaneously, while $\eta_s = 1$ when

* These results have also been obtained for a transformation T smooth at $\mu = 0$, and for a transformation T having, at $\mu = 0$, a discontinuity of the first derivative with respect to x at the point $x = 0$.

$i_s > 1$ and $\eta_s = 0$ when $i_s = 1$. We shall denote a simple fixed point (of multiplicity 1) of the transformation $T_{i_1 j_1 \dots i_N j_N}$ by $x^*_{i_1 j_1 \dots i_N j_N}$, and the cycle containing this fixed point by $C_{i_1 j_1 \dots i_N j_N}$. The fixed point $x^*_{i_1 j_1 \dots i_N j_N}$ is a fixed point of the transformation T of multiplicity $r_N = I_N i_N + J_N j_N$, where

$$I_{k+1} = I_k i_k + J_k j_k, \qquad J_{k+1} = I_k (i_k - \eta_k) + J_k (j_k + 1 - \eta_k)$$

$(k = 1, 2, \dots, N-1)$, $I_1 = J_1 = 1$. By x_1^* and x_2^* we shall denote the simple fixed points of the transformations T_1 and T_2, respectively.

The results of the study are set forth in the following theorems.

Theorem 1. Suppose that for $\mu > 0$, $a(\mu) > 0$, $b(\mu) < 0$, and one of the conditions is satisfied:
1) $0 < f'_x(0,0) < 1,\ 0 < \varphi'_x(0,0) < 1$;
2) $0 < f'_x(0,0) < 1,\ \varphi'_x(0,0) > 1,\ b < a - \varphi(\mu, a)$;
3) $f'_x(0,0) > 1,\ 0 < \varphi'_x(0,0) < 1,\ a < b - f(\mu, a)$.

Then there exists a $\mu_0 > 0$ such that for all $\mu \in (0, \mu_0)$ the transformation T can have a unique stable cycle of type $C_{i_1 j_1 \dots i_N j_N}$ $(N = 1, 2, \dots)$.

Theorem 2. Suppose $a(\mu) > 0,\ b(\mu) < 0$ for $\mu > 0$, and one of the conditions is satisfied:
1) $f'_x(0,0) < 0,\ 0 < \varphi'_x(0,0) < 1$;
2) $0 < f'_x(0,0) < 1,\ \varphi'_x(0,0) < 0$;
3) $f'_x(0,0) < 0,\ \varphi'_x(0,0) > 1,\ b < a - \varphi(\mu, a)$;
4) $f'_x(0,0) > 1,\ \varphi'_x(0,0) < 0,\ a < b - f(\mu, b)$.

Then there exists a $\mu_1 > 0$ such that for all $\mu \in (0, \mu_1)$ the transformation T can have no more than two stable cycles. If conditions 1) or 3) are satisfied, these are the cycles $C_{1n}, C_{1,n+1}$, while if conditions 2) or 4) are satisfied, these are the cycles C_{n1} and $C_{n+1,1}$ $(n = 1, 2, \dots)$.

Denote by $b = A(\mu, a)$ the function given implicitly in the form

$$a + f(\mu, b) = b + \varphi(\mu, a + f(\mu, b)),$$

and by $a = B(\mu, b)$ the function given implicitly in the form

$$a + f(\mu, b + \varphi(\mu, a)) - b - \varphi(\mu, a) = 0.$$

Theorem 3. Suppose the transformation T, for $\mu > 0$, has no stable fixed points, $a(\mu) > 0$, $b(\mu) < 0$, and one of the conditions is satisfied:
1) $0 < f'_x(0,0) < 1,\ 0 < \varphi'_x(0,0) < 1$;
2) $f'_x(0,0) < 0,\ 0 < \varphi'_x(0,0) < 1$;
3) $0 < f'_x(0,0) < 1,\ \varphi'_x(0,0) < 0$;
4) $0 < f'_x(0,0) < 1,\ \varphi'_x(0,0) > 1,\ b < a - \varphi(\mu, a)$;
5) $0 < \varphi'_x(0,0) < 1,\ f'_x(0,0) > 1,\ a < b - f(\mu, b)$;
6) $f'_x(0,0) < 0,\ \varphi'_x(0,0) > 1,\ b < A(\mu, a)$;
7) $f'_x(0,0) > 1,\ \varphi'_x(0,0) < 0,\ a < B(\mu, b)$.

Then there exist a $\mu_2 > 0$ and a domain G on the x-axis such that, for all $\mu \in (0, \mu_2)$, $T^p x \in G$ for any natural p, if $x \in G$.

Theorem 4. Suppose that for $\mu > 0$, $a(\mu) > 0,\ b(\mu) < 0,\ \varphi'_x(0,0) < -1$, and one of the conditions is satisfied:
1) $-1 < f'_x(0,0) < 0,\ b < a - \varphi(\mu, a),\ a < b - f(\mu, b + \varphi(\mu, b))$;
2) $0 < f'_x(0,0) < 1,\ b > a - \varphi(\mu, a),\ a > -f(\mu, b + \varphi(\mu, b))$;
3) $0 < f'_x(0,0) < 1,\ b > a - \varphi(\mu, a),\ a < -f(\mu, b + \varphi(\mu, b))$.

Then there exists a $\mu_3 > 0$ such that for all $\mu \in (0, \mu_3)$ the transformation T can have a unique stable cycle of type $C_{i_1 j_1 \dots i_N j_N}$ $(N = 1, 2, \dots)$.

If condition 1) is satisfied, $i_1 = 1$, and C_{1j_1} is a cycle containing a simple fixed point of the transformation

$$T_{1j_1} = T_1 T_2^{2j_1+1} \quad (j_1 = 0, 1, 2, \dots);$$

if condition 2) is satisfied, $i_1 = 1$, and C_{1j_1} is a cycle containing a simple fixed point of the transformation

$$T_{1j_1} = T_1 T_2^{2j_1} \quad (j = 1, 2, \dots);$$

if condition 3) is satisfied, $j_1 = 2,\ i_1 = 1, 2, \dots$, and $C_{i_1 2}$ is a cycle containing a simple fixed point of the transformation

$$T_{i_1 2} = T_1^{i_1} T_2^2.$$

Theorem 5. Suppose that for $\mu > 0$, $a(\mu) < 0,\ b(\mu) < 0,\ f'_x(0,0) < -1$, and one of the conditions is satisfied:
1) $-1 < \varphi'_x(0,0) < 0,\ a < b - f(\mu, b),\ b < a - \varphi(\mu, a + f(\mu, a))$;

2) $0 < \varphi'_x(0,0) < 1,\ a > b - f(\mu, b),\ b > -\varphi(\mu, a + f(\mu, a))$;
3) $0 > \varphi'_x(0,0) < 1,\ a > b - f(\mu, b),\ b < -\varphi(\mu, a + f(\mu, a))$.

Then there exists a $\mu_4 > 0$ such that, for all $\mu \in (0, \mu_4)$, the transformation T can have a unique stable cycle of type $C_{i_1 \dots i_N j_N}$ $(N = 1, 2, \dots)$. If condition 1) is fulfilled, $j_1 = 1$, and $C_{i_1 1}$ is a cycle containing the simple fixed point of the transformation

$$T_{i_1} = T_1^{2i_1+1} T_2 \quad (i_1 = 0, 1, 2, \dots);$$

if condition 2) is fulfilled, $j_1 = 1$, and $C_{i_1 1}$ is a cycle containing the simple fixed point of the transformation

$$T_{i_1} = T_1^{2i_1} T_2 \quad (i_1 = 1, 2, \dots);$$

if condition 3) is fulfilled, $i_1 = 2,\ j_1 = 1, 2, \dots$, and C_{2j_1} is a cycle containing the simple fixed point of the transformation

$$T_{2j_1} = T_1^2 T_2^{j_1}.$$

Denote by $a = C(\mu, b)$ the function given implicitly in the form

$$a + f(\mu, b + \varphi(\mu, b)) = b + \varphi(\mu, a + f(\mu, b + \varphi(\mu, b))),$$

and by $b = D(\mu, a)$ the function given implicitly in the form

$$b + \varphi(\mu, a + f(\mu, a)) = a + f(\mu, b + \varphi(\mu, a + f(\mu, a))).$$

Theorem 6. Suppose that for $\mu > 0$ either $a(\mu) > 0,\ b(\mu) > 0,\ \varphi'_x(0,0) < -1$, and one of the conditions is fulfilled: 1) $-1 < f'_x(0,0) < 0,\ b > a - \varphi(\mu, a),\ a < C(\mu, b)$; 2) $0 < f'_x(0,0) < 1,\ a < b < a - \varphi(\mu, a),\ a > C(\mu, b)$; or $a(\mu) < 0,\ b(\mu) < 0,\ f'_x(0,0) < -1$, and one of the conditions is fulfilled: 3) $-1 < \varphi'_x(0,0) < 0,\ a > b - f(\mu, b),\ b < D(\mu, a)$; 4) $0 < \varphi'_x(0,0) < 1,\ b < a < b - f(\mu, b),\ b > D(\mu, a)$.

Then there exists a $\mu_5 > 0$ such that, for all $\mu \in (0, \mu_5)$, the transformation T can have no more than two stable cycles. These are the cycles $C_{1,2n}, C_{1,2n+2}$ when condition 1) is fulfilled; the cycles $C_{2n,1}, C_{2n+2,1}$ when condition 2) is fulfilled; the cycles $C_{1,2n-1}, C_{1,2n+1}$ when condition 3) is fulfilled; and the cycles $C_{2n-1,1}, C_{2n+1,1}$ when condition 4) is fulfilled $(n = 0, 1, 2, \dots)$.

Theorem 7. Suppose that for $\mu > 0$ either $a(\mu) > 0,\ b(\mu) > 0,\ 0 < f'_x(0,0) < 1,\ \varphi'_x(0,0) < -1$, and one of the conditions is fulfilled: 1) $a > b,\ a < -f(\mu, b + \varphi(\mu, a))$; 2) $a < b < a - \varphi(\mu, a),\ a < -f(\mu, b + \varphi(\mu, a))$; or $a(\mu) < 0,\ b(\mu) < 0,\ f'_x(0,0) < -1,\ 0 < \varphi'_x(0,0) < 1$, and one of the conditions is fulfilled: 3) $b > a,\ b < -\varphi(\mu, a + f(\mu, b))$; 4) $b < a < b - f(\mu, b),\ b < -\varphi(\mu, a + f(\mu, a))$.

Then there exists a $\mu_6 > 0$ such that, for all $\mu \in (0, \mu_6)$, the transformation T can have a unique stable cycle. In the first two cases this is C_{m1}, and in the remaining cases C_{1m} $(m = 1, 2, \dots)$.

3. In the general case, the transformation T of the line into the line depends on the parameters $\sigma_1, \sigma_2, \ldots, \sigma_m$. Bifurcations of fixed points of a sufficiently smooth transformation of the line into the line have been studied in ([11]) and occur when passing through the bifurcation surfaces N_{+1}, N_{-1}, and also through the surfaces corresponding to the arrival of a fixed point of the transformation T at the boundary of the domain of definition of this transformation.

In the case of a discontinuous transformation of the line into the line with a single discontinuity, new types of bifurcation surfaces have been found. Among them, we first point out those bifurcation surfaces on one side of each of which there is a domain of existence of a unique cycle of stable fixed points of the transformation T, while on the other side, in an arbitrarily thin layer, there is a countable set of pairwise nonintersecting domains of existence of various stable cycles of this transformation and an uncountable set of domains of existence of the corresponding domains G.

For other types of bifurcation surfaces the following is characteristic: on one side of each of them there is a domain of existence of a unique cycle of stable fixed points of the transformation T, while on the other side there is located: 1) either a domain of existence of two different stable cycles of the transformation T; 2) or a domain of existence of the domain G; 3) or a domain for which the transformation T has neither stable cycles nor a domain G. Bifurcation surfaces have also been found which separate the domain of existence of the domain G from the domain of absence, for the transformation T, of both stable cycles and a domain G.

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Note: Figure translations are in progress. See original paper for figures.