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ON THE THEORY OF RADICALS IN GROUPS

The present note is a direct continuation of the work of A. G. Kurosh (¹) on the theory of radicals in groups.

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Abstract

Full Text

MATHEMATICS

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ON THE THEORY OF RADICALS IN GROUPS

(Presented by Academician P. S. Aleksandrov, 11 X 1961)

The present note is a direct continuation of the work of A. G. Kurosh ⁽¹⁾ on the theory of radicals in groups.

§ 1. Let Σ denote, everywhere where this is not specifically stated otherwise, an arbitrary abstract class of groups containing the unit group.

In an arbitrary group G the following conditions are equivalent:

- (b) The product of invariant Σ -subgroups of G is a Σ -group.
- (b₁) The subgroup of the group G generated by the Σ -subgroups attainable in G is a Σ -group.

Obviously, (b) follows from (b₁). The converse assertion is easily derived from the following fact.

Lemma 1. If Σ is a class of groups with condition (b), then every attainable Σ -subgroup of a group G generates an invariant Σ -subgroup in this group.

From this lemma and axiom II.21 of ⁽¹⁾ we obtain also the following

Corollary 1. If R is a radical class, then a group G in which every homomorphic image different from the unit group contains a nonunit attainable R -subgroup is an R -group.

It is also easy to see that axiom I.2 from ⁽¹⁾ for Σ is equivalent to condition (b) and, consequently, to (b₁). In the case of a radical class R this leads to the following:

In every group G its R -radical coincides with the subgroup generated by all attainable R -subgroups in G .

Let us note, finally, that the system of radicality axioms I.1–I.3 of A. G. Kurosh ⁽¹⁾ for a class of groups R is equivalent to the following:

- I.a. A homomorphic image of an R -group is an R -group.
- I.b. The product of invariant R -subgroups of a group G is an R -group.
- I.c. An extension of an R -group by means of an R -group is an R -group.

§ 2. Denote, following ⁽¹⁾, by M a class of groups satisfying axiom I.a, and by $R_0(M)$ the minimal radical class containing all M -groups. Below are given some characteristics of $R_0(M)$ -groups.

Theorem 1. A group G is an $R_0(M)$ -group if and only if every homomorphic image of it different from the unit group contains a nonunit attainable M -subgroup.

The necessity of this assertion is trivial in view of ⁽¹⁾, and the sufficiency is easily derived from Corollary 1 of the preceding paragraph.

A. G. Kurosh established ⁽¹⁾ that the groups of α -th degree over M , where α is an arbitrary ordinal number, and only they, constitute the class $R_0(M)$.

Theorem 2. Let α be an infinite ordinal number. Then every group of α -th degree over M has ω_0 -th degree over M , where ω_0 is the first infinite ordinal number.

The last result is a consequence of Theorem 1 and the following lemma:

Lemma 2. Let the class of groups M satisfy condition I.a. Then every attainable M -subgroup $A \neq E$ of defect $d \neq 0$ in a group G generates in this group an invariant subgroup of d -th degree over M .

Let us note that the defect of an attainable subgroup is understood here in the sense of R. Baer ⁽²⁾ and is always a nonnegative integer.

As a consequence of Theorem 2 we obtain still another characterization of the class of groups that interests us.

A group G is an $R_0(M)$ -group if and only if it has an ascending invariant series with factors having finite degree over M .

§ 3. In the work of R. Baer ⁽²⁾, nilgroups were studied, i.e., groups generated by their ascendant cyclic subgroups. It is useful to generalize this concept to the case of an arbitrary abstract class Σ containing the identity group.

Definition. We shall call a group G a Σ -nilgroup if it is generated by its ascendant Σ -subgroups.

From the transitivity of the property of being an ascendant subgroup it is easy to obtain that Σ -nilgroups satisfy condition (b) and, consequently, (b_1) . Hence in every group G there exists a characteristic Σ -nilsubgroup $\Sigma(G)$, generated by all ascendant Σ -nilsubgroups in G . Moreover, $\Sigma(G)$ coincides with the subgroup of the group G generated by all ascendant Σ -subgroups in G . As follows from § 1, in the case of a radical class Σ the subgroup $\Sigma(G)$ coincides with the Σ -radical of the group G .

If now the class M satisfies condition I.a, then this condition will obviously also be satisfied by the class of M -nilgroups. Hence, from Theorem 1, it is clear that M -nilgroups are contained in the class of $R_0(M)$ -groups and, generally speaking, properly so ⁽²⁾.

Construct in an arbitrary group G an ascending characteristic series

$$E = M_0(G) \subset M_1(G) \subset \dots \subset M_\alpha(G) \subset M_{\alpha+1}(G) \subset \dots$$

according to the rule: $M(G/M_\alpha(G)) = M_{\alpha+1}(G)/M_\alpha(G)$, and at limit positions take, as usual, the union of the preceding terms. There exists an ordinal λ such that $M_\lambda(G) = M_{\lambda+1}(G)$. It turns out that the subgroup $M_\lambda(G)$ coincides with the $R_0(M)$ -radical of the group G . Hence there arises still another characterization of the class $R_0(M)$:

A group G is an $R_0(M)$ -group if and only if it has an ascending characteristic series with factors that are M -nilgroups.

§ 4. According to Theorem 2, the first infinite ordinal serves as an upper bound for the degree over M of every $R_0(M)$ -group. Here we consider the simplest condition under which a natural number serves as such a bound.

Let us first consider the following condition for an abstract class of groups Σ .

(β) If a group $A \neq E$, generated by its invariant Σ -subgroups, is a normal divisor of a group B , then A contains a nonidentity Σ -subgroup invariant in the whole group B .

Lemma 3. *If condition (β) is fulfilled for Σ , then every nonidentity ascendant Σ -subgroup of minimal defect in a group G is a normal divisor of this group.*

From Theorem 1 and Lemma 3 we easily obtain the following assertion.

If the class of groups of degree n over M , where $n \geq 1$ is a fixed natural number, satisfies condition (β), then the minimal radical class $R_0(M)$ consists exactly of the groups of degree n over M .

Let us recall ⁽¹⁾, by the way, that the groups of degree 2 over the class of abelian groups coincide with the RJ^* -groups ⁽³⁾, while the subresolvable groups of R. Baer ⁽²⁾ are exactly the minimal radical class over the class of abelian groups (see Theorem 1). Up to now the question ⁽²⁾ of the coincidence of these two classes of groups has remained open. It is even more unknown whether abelian groups will satisfy condition (β).

Intermediate between (b) and (β) for Σ is the following condition (cf. ⁽²⁾):

(β_0) In every group G the product of all nontrivial invariant Σ -subgroups contains a nontrivial characteristic Σ -subgroup.

Since in the presence of (β_0) condition (β) is fulfilled automatically, from the preceding result we obtain:

If the class of groups of the n -th degree over M , where $n \geq 1$ is a fixed natural number, satisfies condition (β_0), then the class $R_0(M)$ consists of the groups that possess, in every nontrivial homomorphic image, a nontrivial characteristic subgroup of the n -th degree over M .

R. Baer ⁽²⁾ posed the question of the fulfillment of (β_0) for abelian groups. D. McLain ⁽⁴⁾ published an example of an elementary locally finite p -group which is an RJ^* -group, thereby giving a negative answer to this question.

§ 5. Recall that a subinvariant subgroup of a group G is any subgroup of it that is a member of some ascending (in general, transfinite) normal series of the group G .

Let the abstract class of groups Σ be subject to the requirement:

(*) The set-theoretic sum of an ascending normal chain of Σ -subgroups of a group G is a Σ -group.

Then, in an arbitrary group G , condition (b) for such a Σ is equivalent to the following:

(b_2) *The subgroup of the group G generated by the subinvariant in G Σ -subgroups is a Σ -group.*

It is obvious that (b) follows from (b_2). For the proof of the converse assertion, one uses

Lemma 4. *Let the abstract class Σ , with condition (*), be subject to requirement (b). Then every subinvariant Σ -subgroup of the group G generates in this group an invariant Σ -subgroup.*

The last lemma and Axiom II.21 from ⁽¹⁾ also allow one to obtain

Corollary 2. *If R is a radical class of groups with condition (*), then a group G such that in every nontrivial homomorphic image of it there is a nontrivial subinvariant R -subgroup is an R -group.*

As already noted in § 1, Axiom I.2 from ⁽¹⁾ is equivalent to condition (b) and, consequently, in the presence of (*), to condition (b_2). Hence we obtain:

If a radical class R satisfies condition (), then the R -radical of a group G coincides with the subgroup generated by all subinvariant in G R -subgroups.*

Let, as in ⁽¹⁾, $R(M)$ denote the class of groups possessing an ascending normal series with M -factors. The class of $R(M)$ -groups is radical ⁽¹⁾ and, obviously, is subject to requirement (*). Hence, also from Corollary 2, it is easily derived that

Theorem 3. *A group G is an $R(M)$ -group if and only if every nontrivial homomorphic image of it contains a nontrivial subinvariant M -subgroup.*

In conclusion I express my sincere gratitude to Prof. A. G. Kurosh, under whose supervision this work was carried out.

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Note: Figure translations are in progress. See original paper for figures.

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