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Abstract

Full Text

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ON THE THEORY OF OPERATIONAL CALCULUS

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V. A. Ditkin (^{1}) constructed an operational calculus for the operator

$$B = \frac{d}{dt}\,t\,\frac{d}{dt},$$

generated by Bessel' s equation

$$y'' + \frac{1}{x}y' - \left(1 - \frac{\nu^2}{x^2}\right)y = 0. \tag{1}$$

In the present note an operational calculus is set forth for the operator

$$T = \frac{d}{dt}\,t\,\frac{d}{dt}\,t\,\frac{d}{dt}.$$

This operator is closely connected with the equation

$$y''' + \frac{3}{x}y'' + \frac{1 - 3m^2 + 3mn - 3n^2}{x^2}y' + \left[1 + \frac{(m+n)(2m-n)(m-2n)}{x^3}\right]y = 0. \tag{2}$$

If in equation (2) $m = n = 0$, then we shall have

$$y''' + \frac{3}{x}y'' + \frac{1}{x^2}y' + y = 0. \tag{3}$$

Put $x = 3\sqrt[3]{\lambda t}$. Then the preceding equation takes the form

$$\frac{d}{dt}\,t\,\frac{d}{dt}\,t\,\frac{d}{dt}y + \lambda y = 0. \tag{4}$$

Denote by L_T the set of all functions $f(t)$, defined on the half-line $0 \leq t < \infty$, Lebesgue integrable on every finite interval $(0, A)$, and satisfying the condition

$$\int_0^t \frac{dx}{x} \int_0^1 \frac{dy}{y} \int_0^{xy} |f(t)|\, dt < \infty$$

for every $t_0 > 0$. Denote by M_T the set of all functions of the form

$$F(t) = \int_0^t \frac{dx}{x} \int_0^1 \frac{dy}{y} \int_0^{xy} f(t)\, dt + C,$$

where $f(t)$ is an arbitrary function from L_T, and C is an arbitrary constant. In the set L_T define the product of functions $f_1(t) \in L_T$ and $f_2(t) \in L_T$ by the formula

$$f(t) = \int_0^t d\tau \int_0^1 dx \int_0^1 f_1(xy\tau)\, f_2[(t-\tau)(1-x)(1-y)]\, dy.$$

The function $f(t)$ belongs to the set L_T. For every function of the set M_T there exists almost everywhere a third derivative $F'''(t)$. Define in the set M_T the operation of multiplication by the formula

$$F_1(t) * F_2(t) = \frac{d}{dt}\, t\, \frac{d}{dt}\, t\, \frac{d}{dt} \int_0^t d\tau \int_0^1 dx \int_0^1 F_1(xy\tau)\, F_2[(t-\tau)(1-x)(1-y)]\, dy. \tag{5}$$

If

$$F_i(t) = \int_0^t \frac{dx}{x} \int_0^1 \frac{dy}{y} \int_0^{xy} f_i(t)\, dt \qquad (i = 1, 2),$$

$$f(t) = \int_0^t d\tau \int_0^1 dx \int_0^1 f_1(xy\tau)\, f_2[(t-\tau)(1-x)(1-y)]\, dy,$$

then we shall have

$$F_1(t) * F_2(t) = \int_0^t \frac{dx}{x} \int_0^1 \frac{dy}{y} \int_0^{xy} f(t)\, dt.$$

Addition in the set M_T is defined in the natural way. The product defined in (5) again belongs to the set M_T; it is commutative, associative, and has distributivity with respect to addition. Consequently, the set M_T forms a commutative ring. Moreover, if one of the factors in (5) is a number α, then we have

$$\alpha f_1(t) = \alpha \frac{d}{dt} t \frac{d}{dt} t \frac{d}{dt} \int_0^t d\tau \int_0^1 dx \int_0^1 f_1[(t-\tau)(1-x)(1-y)]\,dy =$$

$$= \alpha \frac{d}{dt} t \frac{d}{dt} t \frac{d}{dt} \int_0^t \frac{d\tau}{\tau} \int_0^1 \frac{dx}{x} \int_0^{x\tau} f_1(\xi)\,d\xi =$$

$$= \alpha \frac{d}{dt} t \frac{d}{dt} \int_0^t \frac{du}{u} \int_0^u f_1(\xi)\,d\xi = \alpha B \frac{1}{B} f_1(t) = \alpha f_1(t),$$

i.e., the product of a number by a function in the ring coincides with the ordinary product of a number by a function. If in (5) both factors are numbers, then the product (5) coincides with the ordinary product of numbers.

It can be proved that the ring M_T has no zero divisors. Consequently, the ring M_T can be extended to a field of quotients. We shall denote the extended ring by $\mathfrak{M}_T$. The elements of the set $\mathfrak{M}_T$ will be called operators. For the operator $\frac{1}{t}$ we introduce the notation $\frac{1}{t} = T$. Then for the inverse operator $T^{-1} = \frac{1}{T}$ we shall have $\frac{1}{T} = t$. Therefore

$$\frac{1}{T} f(t) = \frac{d}{dt} t \frac{d}{dt} t \frac{d}{dt} \int_0^t d\tau \int_0^1 dx \int_0^1 xy\tau f[(t-\tau)(1-x)(1-y)]\,dy,$$

or

$$\frac{1}{T} f(t) = \int_0^t \frac{d\tau}{\tau} \int_0^1 \frac{dx}{x} \int_0^{x\tau} f(\xi)\,d\xi. \tag{6}$$

If $F(t) \in M_T$ and $F(0) = 0$, then

$$TF(t) = \frac{d}{dt} t \frac{d}{dt} t \frac{d}{dt} F(t) = t^2 F'''(t) + 3tF''(t) + F'(t). \tag{7}$$

From (6) it follows that

$$\frac{1}{T^n} f(t) = \frac{1}{(n!)^3} \int_0^t (t-\tau)^n\,d\tau \int_0^1 (1-x)^n\,dx \int_0^1 f(xy\tau)(1-y)^n\,dy. \tag{8}$$

In the particular case we have

$$\frac{1}{T^n} = \frac{t^n}{(n!)^3}. \tag{9}$$

One of the linearly independent solutions of equation (4), namely

$$J_{0,0}^{(2)}(3\sqrt[3]{\lambda t}) = {}_0F_2(1,1;-\lambda t),$$

for $t = 0$ takes the value $J_{0,0}^{(2)}(0) = 1$. Therefore from (4), (6), (7) it follows that

$$T\left[J_{0,0}^{(2)}(-3\sqrt[3]{\lambda t}) - J_{0,0}^{(2)}(0)\right] = \frac{d}{dt}t\frac{d}{dt}t\frac{d}{dt}J_{0,0}^{(2)}(-3\sqrt[3]{\lambda t})$$

or

$$T\left[J_{0,0}^{(2)}\left(-3\sqrt[3]{\lambda t}\right) - 1\right] = \lambda J_{0,0}^{(2)}\left(-3\sqrt[3]{\lambda t}\right).$$

Hence

$$\frac{T}{T+\lambda} = J_{0,0}^{(2)}\left(3\sqrt[3]{\lambda t}\right), \qquad \frac{T}{T-\lambda} = I_{0,0}^{(2)}\left(3\sqrt[3]{\lambda t}\right), \tag{10}$$

where

$$I_{0,0}^{(2)}(x) = {}_0F_2\left[1,1;\left(\frac{x}{3}\right)^3\right].$$

With the aid of formulas (10) it is easy to obtain

$$\frac{T^2}{T^2-\lambda^2} = \frac{1}{2}\left[I_{0,0}^{(2)}\left(3\sqrt[3]{\lambda t}\right) + J_{0,0}^{(2)}\left(3\sqrt[3]{\lambda t}\right)\right]; \tag{11}$$

$$\frac{\lambda T}{T^2-\lambda^2} = \frac{1}{2}\left[I_{0,0}^{(2)}\left(3\sqrt[3]{\lambda t}\right) - J_{0,0}^{(2)}\left(3\sqrt[3]{\lambda t}\right)\right]; \tag{12}$$

$$\frac{T^2}{T^2+\omega^2} = \mathrm{ber}_{0,0}^{(2)}\left(3\sqrt[3]{\omega t}\right), \qquad \frac{\omega T}{T^2+\omega^2} = \mathrm{bei}_{0,0}^{(2)}\left(3\sqrt[3]{\omega t}\right), \tag{13}$$

where

$$\mathrm{ber}_{0,0}^{(2)}(x) = \sum_{k=0}^{\infty}\frac{(-1)^k(x/3)^{6k}}{[(2k)!]^3}, \qquad \mathrm{bei}_{0,0}^{(2)}(x) = \sum_{k=0}^{\infty}\frac{(-1)^k(x/3)^{3(2k+1)}}{[(2k+1)!]^3}.$$

Applying to the field of operators $\mathfrak{M}_T$ the known operational methods ([2],[3]), one can substantially extend the table of values of the operators (10), (11), (12), (13). For example, differentiating (10) with respect to the parameter λ and then putting $\lambda = 1$ and $\lambda = -1$, we obtain

$$\frac{T}{(T+1)^{n+1}} = \frac{1}{n!}\, t^{n/3} J^{(2)}_{n,n}\left(3\sqrt[3]{t}\right); \tag{14}$$

$$\frac{T}{(T-1)^{n+1}} = \frac{1}{n!}\, t^{n/3} I^{(2)}_{n,n}\left(3\sqrt[3]{t}\right), \tag{15}$$

where

$$J^{(2)}_{n,n}(x) = \frac{(x/3)^{2n}}{\Gamma^2(n+1)}\, {}_0F_2\left[n+1, n+1; -\left(\frac{x}{3}\right)^3\right],$$

$$I^{(2)}_{n,n}(x) = \frac{(x/3)^{2n}}{\Gamma^2(n+1)}\, {}_0F_2\left[n+1, n+1; \left(\frac{x}{3}\right)^3\right].$$

From (13) we find

$$T\left(\frac{\omega T}{T^2+\omega^2}\right) = \frac{\omega T^2}{T^2+\omega^2} = \omega\ \mathrm{ber}^{(2)}_{0,0}\left(3\sqrt[3]{\omega t}\right),$$

$$T\left(\frac{T^2}{T^2+\omega^2} - 1\right) = -\frac{\omega^2 T}{T^2+\omega^2} = -\omega\left(\frac{\omega T}{T^2+\omega^2}\right) = -\omega\ \mathrm{bei}^{(2)}_{0,0}\left(3\sqrt[3]{\omega t}\right).$$

Hence it follows that

$$\frac{d}{dt}\, t\, \frac{d}{dt}\, t\, \frac{d}{dt}\ \mathrm{bei}^{(2)}_{0,0}\left(3\sqrt[3]{\omega t}\right) = \omega\ \mathrm{ber}^{(2)}_{0,0}\left(3\sqrt[3]{\omega t}\right),$$

$$\frac{d}{dt}\, t\, \frac{d}{dt}\, t\, \frac{d}{dt}\ \mathrm{ber}^{(2)}_{0,0}\left(3\sqrt[3]{\omega t}\right) = -\omega\ \mathrm{bei}^{(2)}_{0,0}\left(3\sqrt[3]{\omega t}\right).$$

Multiplying equality (15) by μ^n and summing with respect to n from 0 to ∞, we find

$$\sum_{n=0}^{\infty} \frac{T\mu^n}{(T-1)^{n+1}} = \frac{T}{T-1-\mu},$$

whence

$$\sum_{n=0}^{\infty} \frac{\mu^n}{n!}\, t^{n/3} I^{(2)}_{n,n}\left(3\sqrt[3]{t}\right) = I^{(2)}_{0,0}\left(3\sqrt[3]{(1+\mu)t}\right).$$

The operational calculus for the operator $T = \frac{d}{dt}\,t\,\frac{d}{dt}\,t\,\frac{d}{dt}$ can be constructed starting from the corresponding integral transform. The analogue of the Laplace transform here will be the integral transform

$$f^*(T) = 2\int_0^\infty f(t)\,M(Tt)\,dt,$$

where

$$M(t) = 2\int_0^\infty e^{-4t/u^2} K_0(u)\,\frac{du}{u}$$

is the solution of equation (4) for $\lambda = 1$. The operational calculus for the operator

$$T = \frac{d}{dt}\,t\,\frac{d}{dt}\,t\,\frac{d}{dt}$$

can also be applied to the solution of differential equations. The solutions of differential equations of the form

$$L\left(\frac{d}{dt}\,t\,\frac{d}{dt}\,t\,\frac{d}{dt}\right)x(t) = f(t),$$

where $L(\lambda) = \lambda^n + a_1\lambda^{n-1} + \dots + a_n$ is a polynomial with constant coefficients a_i, are found most simply. Replacement of the operator $\frac{d}{dt}\,t\,\frac{d}{dt}\,t\,\frac{d}{dt}$ by the operator T is carried out according to the formula

$$\left(\frac{d}{dt}\,t\,\frac{d}{dt}\,t\,\frac{d}{dt}\right)^n x(t) = T^n x(t) - Tx_{n-1} - T^2 x_{n-2} - \dots - T^n x_0,$$

where

$$x_k = T^k x(t)\big|_{t=0} \qquad (k = 0, 1, 2, \dots, n-1).$$

For example, let us find the solution of the equation

$$\left(\frac{d}{dt}\,t\,\frac{d}{dt}\,t\,\frac{d}{dt}\right)^2 x(t) + x(t) = 0$$

under the condition

$$x(0) = 1, \qquad Tx(t)\big|_{t=0} = \left[t^2 x'''(t) + 3tx''(t) + x'(t)\right]_{t=0} = 1.$$

After replacing the operator $\frac{d}{dt}\,t\,\frac{d}{dt}\,t\,\frac{d}{dt}$ by the operator T, we shall have

$$T^2 x(t) - T - T^2 + x(t) = 0,$$

whence

$$x(t) = \frac{T^2}{T^2+1} + \frac{T}{T^2+1} = \mathrm{ber}_{0,0}^{(2)}\left(3\sqrt[3]{t}\right) + \mathrm{bei}_{0,0}^{(2)}\left(3\sqrt[3]{t}\right).$$

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Note: Figure translations are in progress. See original paper for figures.

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