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Abstract

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MATHEMATICS

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AN EXAMPLE OF A FUNCTION HARMONIC IN ALL SPACE AND BOUNDED OUTSIDE A CIRCULAR CYLINDER

(Presented by Academician M. V. Keldysh on 23 X 1961)

The present note, like the notes ($^{1-3}$), is devoted to a problem that may be formulated as follows.

Let D be some domain in the plane (x_1, x_2), and let $u(x, x_1, x_2)$ be a function harmonic in the half-cylinder

$$(x_1, x_2) \in D, \qquad x > 0. \tag{1}$$

Find $\sigma(D)$, the lower bound of those values σ for which it follows from conditions A and B that $u(x, x_1, x_2)$ is bounded also inside the half-cylinder (1):

A.

$$\max_{(x_1,x_2)\in D} |u(x, x_1, x_2)| < M \exp\exp\frac{\pi x}{\sigma} \quad (x > 0).$$

B. $u(x, x_1, x_2)$ and $\operatorname{grad} u(x, x_1, x_2)$ are bounded on the surface of the half-cylinder (1).

The analogous problem for functions harmonic in the plane is easily reduced to the Phragmén–Lindelöf theorem on functions analytic in a strip.

In the notes (^{1,2}) the constant $\sigma(D)$ was found for the case when D is a circle or a rectangle, and in the note (^{3}) an upper estimate for $\sigma(D)$ was obtained for the case of an arbitrary domain. To formulate the results, we introduce two geometric characteristics of the domain. By the **external width** $H(D)$ of the domain D we shall mean the width of the narrowest strip into which this domain can be placed. By the **internal width** $h(D)$ of the domain D we shall mean the diameter of the largest circle that can be placed in this domain.

The result of the note (^{3}) consists, roughly speaking, in the fact that

$$\sigma(D) \leq H(D),$$

and in the present note it will be proved that

$$\sigma(D) \geq h(D).$$

Thus, for domains for which $H(D)$ and $h(D)$ coincide (in particular, for a circle, a rectangle, and a semicircle), the constant $\sigma(D)$ is equal to their common value.

To prove the inequality $\sigma(D) \geq h(D)$, it is sufficient to construct a function harmonic in all space, bounded outside a circular half-cylinder with base radius $\pi\lambda/2$, and inside it growing, but more slowly than $\exp\exp(x/\lambda_1)$ with any $\lambda_1 < \lambda$ (as $x \to +\infty$). The boundedness of the gradient of such a function outside our half-cylinder will be obtained from the boundedness of the function itself by virtue of the Poisson formula.

The construction of an example of such a function constitutes the content of the note.

Theorem. The function

$$u_\lambda(x, x_1, x_2) = \int_0^\infty J_0\left(s\sqrt{x_1^2 + x_2^2}\right) e^{xs - \lambda s \ln \lambda s}\, ds, \tag{2}$$

is harmonic in all space, bounded outside the half-cylinder

$$x_1^2 + x_2^2 \leq \left(\frac{\pi\lambda}{2}\right)^2, \qquad x \geq 0,$$

not being bounded inside it, and satisfies the inequality

$$\max_{x_1, x_2} |u_\lambda(x, x_1, x_2)| < M_\varepsilon \exp\exp\frac{x}{\lambda - \varepsilon}, \qquad x > 0,$$

with any $\varepsilon > 0$.

Proof. The integral (2) and all its derivatives converge uniformly for all x, x_1, x_2, and from the differential equation for the Bessel function $J_0(z)$ it follows that $u_\lambda(x, x_1, x_2)$ is harmonic in the whole space.

It is obvious that $u_\lambda(x, 0, 0)$ is unbounded as $x \to +\infty$ and that, for any x_1, x_2, the inequality

$$|u_\lambda(x, x_1, x_2)| \leq u_\lambda(x, 0, 0) < M_\varepsilon \exp\exp\frac{x}{\lambda - \varepsilon} \qquad (x > 0)$$

holds with any $\varepsilon > 0$.

It remains to show that $u_\lambda(x, x_1, x_2)$ is bounded when $x_1^2 + x_2^2 > (\pi\lambda/2)^2$. We shall use the formula

$$J_0(\xi) = \frac{1}{\pi i} \int_1^\infty \frac{e^{it\xi}\, dt}{\sqrt{t^2-1}} - \frac{1}{\pi i} \int_1^\infty \frac{e^{-it\xi}\, dt}{\sqrt{t^2-1}} \qquad (\xi > 0), \tag{3}$$

which is easily obtained from the well-known formula

$$J_0(\xi) = \frac{1}{2\pi i} \int_{|z|=R} \frac{e^{\xi z}\, dz}{\sqrt{z^2+1}} \qquad (R > 1)$$

by deforming the contour of integration.

Substituting formula (3) into formula (2) for u_λ, we obtain

$$u_\lambda(x, \rho\sin\varphi, \rho\cos\varphi) = \frac{1}{\pi i} \int_0^\infty \int_1^\infty \frac{e^{i\rho st + xs - \lambda s \ln \lambda s}}{\sqrt{t^2-1}}\, dt\, ds - \frac{1}{\pi i} \int_0^\infty \int_1^\infty \frac{e^{-i\rho st + xs - \lambda s \ln \lambda s}}{\sqrt{t^2-1}}\, dt\, ds.$$

Rotating the ray of integration $(0, +\infty)$ in the first term by the angle $+\pi/2$, and in the second term by the angle $-\pi/2$, we find

$$u_\lambda = \frac{1}{\pi} \int_0^\infty \int_1^\infty \frac{e^{-\rho t y + ixy - i\lambda \ln(i\lambda y)}}{\sqrt{t^2-1}}\, dt\, dy - \frac{1}{\pi} \int_0^\infty \int_1^\infty \frac{e^{-\rho t y - ixy + i\lambda y \ln(-i\lambda y)}}{\sqrt{t^2-1}}\, dt\, dy,$$

whence

$$|u_\lambda(x, \rho\sin\varphi, \rho\cos\varphi)| \le \frac{2}{\pi} \int_0^\infty \int_1^\infty e^{-\rho t y + \frac{1}{2}\pi\lambda y} \frac{dt}{\sqrt{t^2-1}}\, dy.$$

For $\rho > \pi\lambda/2$ the integral converges, and we obtain

$$|u_\lambda(x, x_1, x_2)| \le \frac{M}{\sqrt{x_1^2 + x_2^2} - \pi\lambda/2}.$$

The theorem is proved.

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References cited

1. M. A. Evgrafov, I. A. Chegis, DAN, **134**, No. 2 (1960).

2. I. A. Chegis, DAN, **136**, No. 3 (1961).

3. I. S. Arshon, M. A. Evgrafov, DAN, **142**, No. 4 (1962).

Note: Figure translations are in progress. See original paper for figures.

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