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Abstract

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MATHEMATICS

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CLASSIFICATION OF POISSON-STABLE MOTIONS. PSEUDORECURRENT MOTIONS

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In the general theory of dynamical systems one distinguishes a number of types of Poisson-stable motions, the classification of which is indicated in ([1]).

In the present note, proceeding from one general definition, various particular cases of which exhaust the definitions of all known types of Poisson-stable motions, a new class of motions is revealed, and its properties and connection with known classes are established.

1°. Let $f(p,t)$ be a dynamical system defined in an arbitrary metric space R ([2]). In what follows an essential role will be played by a certain function $T = T(\varepsilon,t,l)$ of the variables ε, t, and l, varying respectively over the sets $(0,+\infty)$, $(-\infty,+\infty)$, and $[0,+\infty)$. In order to distinguish those cases when T does not depend on some of its arguments, we introduce into consideration the set $\{E\}$ of all subsets of the set $\{t,l\}$, whose elements are the variables t and l.

Definition 1. We shall assign the motion $f(p,t)$ to the **set of motions** Ω^E (A^E), if there exists a nonnegative function $T = T(\varepsilon,t,l)$, defined for all $\varepsilon > 0$, $t \in (-\infty,+\infty)$, and $l \geq 0$, such that:

1) for any triple of numbers ε, t, and l, which are values of the corresponding variables, on the interval $[l, l+T]$ ($[-l-T, -l]$) there exists a number τ such that

$$\rho[f(p,t+\tau), f(p,t)] \leq \varepsilon; \qquad (1)$$

2) the function T does not depend on the variables from E. The number τ satisfying inequality (1) is called an ε-shift of the point $f(p,t)$.

Let us consider all possible sets Ω^E and A^E for possible E, $E \subseteq \{t,l\}$. (As is easy to see, there will be 8 of them.) It turns out that each of the introduced sets, with the exception of $\Omega^t = A^t$, coincides with one or another known class of Poisson-stable motions. This connection is indicated in the following table.

Set of motions	Classes of motions
Ω^{Λ} (A^{Λ})	Stable p^{+} (p^{-})
$\Omega^{l} = A^{l}$	Almost recurrent
$\Omega^{t,l} = A^{t,l}$	Recurrent

Motions belonging to the set Ω^{t} will be called pseudorecurrent. As will be shown below, the set of pseudorecurrent motions contains the set of all uniformly Poisson-stable motions, without coinciding with the latter. Thus, special (i.e., periodic and stationary) and almost periodic motions belong to the set $\Omega^{t,l}$ together with recurrent motions, while uniformly Poisson-stable motions belong to the set Ω^{t} together with pseudorecurrent motions. However, if the quantity τ occurring in Definition 1 is regarded as a function

variables ε, t, and l, then one can obtain a more detailed classification, in which special and almost periodic motions will be separated out from the set of recurrent motions, and uniformly Poisson-stable motions from the set of pseudorecurrent motions. In this case, no new types of Poisson-stable motions will be found.

2°. Let us consider in more detail the set Ω^{t} of pseudorecurrent motions. As is easy to verify, the definition of a pseudorecurrent motion is equivalent to the following. A motion $f(p,t)$ is called pseudorecurrent if, for every pair of positive numbers ε and l, there exists a number L, $L \geq l$, such that on the interval $[l, L]$ there is an ε-shift of every point of the trajectory $f(p, I)$.

It follows from Definition 1 that $\Omega^{t,l} \subseteq \Omega^{t} \subseteq \Omega^{\Lambda} \cap A^{\Lambda}$, i.e. every recurrent motion is pseudorecurrent, and every pseudorecurrent motion is Poisson-stable (in both directions).

Concerning the closure of the trajectory of a pseudorecurrent motion, one can state the following:

Theorem 1. If a point q belongs to the closure of the trajectory of a pseudorecurrent motion, then the motion $f(q,t)$ is pseudorecurrent.

Theorem 2. A motion $f(p,t)$ of a compact dynamical system is pseudorecurrent if and only if, for every point q belonging to the closure of the trajectory $\overline{f(p, I)}$, the motion $f(q,t)$ is stable in the positive direction in the sense of Poisson.

3°. Let us construct an example of a pseudorecurrent motion. Put

$$l_1 = 1; \qquad l_{n+1} = (4n+5)l_n \quad (n = 1, 2, \dots). \tag{2}$$

The segment $[(2i-1)l_n, (2i+1)l_n]$, where i is an integer and n a natural number, will be denoted by σ_i^n. It follows from (2) that the segment σ_i^{n+1} contains an odd number, namely $4n+5$, of segments of the form σ_k^n. Fix an arbitrary n and consider any segment σ_i^{n+1}. Write down all the segments σ_k^n contained in σ_i^{n+1}, in the order in which they occur:

$$\sigma^n_{k_i+1},\quad \sigma^n_{k_i+2},\dots,\quad \sigma^n_{k_i+2n+2},\quad \sigma^n_{k_i+2n+3},\dots,\quad \sigma^n_{k_i+4n+5},$$

where k_i is an integer which can be determined from the equation

$$(2k_i+1)l_n = (2i-1)l_{n+1}. \tag{3}$$

The rightmost segment $\sigma^n_{k_i+4n+5}$ and the segment $\sigma^n_{k_i+2n+2}$ preceding the middle one will be called special segments, and the remaining ones ordinary. If $\sigma^n_{k_i+s}$ is a special segment, then, as is easy to see, $\sigma^n_{k_j+s}$ is also special, while $\sigma^n_{k_j+s+1}$ is ordinary, for every integer j.

Now on $(-\infty,+\infty)$ define a function $\varphi(x)$ so that $\varphi(x) = -1-|x|$ on σ_0^1. Suppose that $\varphi(x)$ is defined on σ_0^n. Then on $\sigma_0^{n+1}\setminus\sigma_1^n$ define $\varphi(x)$ by the formula:

$$\varphi(x) = \begin{cases} \dfrac{n+2-i}{n+2}\,\varphi(x-2il_n) & \text{on } \sigma_i^n \text{ for } i = 1,2,\dots,n+1;\\ 0 & \text{on } \sigma_i^n \text{ for } i = n+2, n+3,\dots,2n+2;\\ \varphi(x+l_n+l_{n+1}) & \text{on } \sigma_i^n \text{ for } i = -2n-2,-2n-1,\dots,-1.\end{cases}$$

Thus, on $(-\infty,+\infty)$, the function $\varphi(x)$ has been constructed by induction. From the definition of this function there follow its properties:

1) $\varphi(x)$ is uniformly continuous and $0\le\varphi(x)\le 1$ on $(-\infty,+\infty)$;

2) if the segment σ_i^n is special, then $\varphi(x)\equiv 0$ on $\sigma_{i-1}^n\cup\sigma_i^n$;

3) if the segment σ_i^n is ordinary and $x\in\sigma_i^n$, then

$$|\varphi(x+2l_n)-\varphi(x)| < \frac{1}{n}.$$

In the dynamical system of M. V. Bebutov (^{1,3}) consider the motion $f(\varphi,t)$, determined by the function $\varphi(x)$.

Lemma. *Let n be a natural number. Then for every positive t there is a number τ, equal to $2l_n$ or $2l_{n+1}$, such that*

$$|\varphi(x+t+\tau)-\varphi(x+t)| < \frac{1}{n} \quad \text{for } x\in[0,2l_n]. \tag{4}$$

Let now ε and l be given positive numbers. Choose a natural n so that $n\ge 1/\varepsilon$ and $2l_n\ge l$, and put $L = 2l_{n+1}$. Let t be any real number. According to the lemma, for the number $t - l_n$, by choosing $\tau = 2l_n$ or $\tau = 2l_{n+1}$, one can ensure that the inequality

$$|\varphi(y + t - l_n + \tau) - \varphi(y + t - l_n)| < \varepsilon$$

holds for $y \in [0, 2l_n]$, whence it follows that

$$|\varphi(x + t + \tau) - \varphi(x + t)| < \varepsilon$$

for $|x| \leq l_n$, and, a fortiori, for $|x| \leq 1/\varepsilon$ (since always $l_n \geq n \geq 1/\varepsilon$). Hence

$$\rho[f(\varphi, t + \tau), f(\varphi, t)] < \varepsilon,$$

with $\tau \in [2l_n, 2l_{n+1}] \subset [l, L]$. According to the definition, the motion $f(\varphi, t)$ is pseudorecurrent.

4°. Obviously, *every uniformly Poisson-stable motion is pseudorecurrent.* However, *even in a compact dynamical system there exists a pseudorecurrent motion which is not uniformly Poisson-stable.*

Indeed, the motion $f(\varphi, t)$, constructed in Sec. 3°, is Lagrange stable in view of property 1 of the function $\varphi(x)$. Let us show that this motion is not uniformly Poisson-stable. Let $\tau \in (1, +\infty)$. Since the sequence $\{l_n\}$ is strictly increasing, and $\tau > 1$, there is a natural n such that $\tau \in \sigma_0^{n+1} \setminus \sigma_0^n$. According to the definition of the function $\varphi(x)$, $\varphi(0) = 1$, while $\varphi(-\tau) = 0$, if

$$\tau \in \bigcup_{i=1}^{n+1} \sigma_i^n.$$

Consequently, the inequality

$$|\varphi(x + \tau) - \varphi(x)| \leq \tfrac{1}{2} \tag{5}$$

is violated at $x = -\tau$. If, however,

$$\tau \in \bigcup_{i=n+2}^{2n+2} \sigma_i^n,$$

then inequality (5) is violated at $x = 0$. Thus it has been shown that the function $\varphi(x)$ is not Bohr pseudoperiodic and, consequently, the motion $f(\varphi, t)$ is not uniformly Poisson-stable.

The motion considered above is also not almost recurrent. Indeed, since there exist intervals of arbitrarily large length on which $\varphi(x) \equiv 0$, while $\varphi(0) = 1$, for $\varepsilon < 1$, for the point $\varphi(x) \in f(\varphi, I)$ there is no relatively dense set of ε-translations.

Thus, *there exists a pseudorecurrent motion which is not almost recurrent (and hence not recurrent).*

The converse is also true: *there exists an almost recurrent motion which is not pseudorecurrent.* An example of such a motion can be constructed in the dynamical system of M. V. Bebutov.

As was already noted, every pseudorecurrent motion is Poisson-stable. However, *even in a compact dynamical system there exists a Poisson-stable motion which is not pseudorecurrent.* One can verify this using the example of the dynamical system given in (^{2}) on page 365, taking into account Theorem 1 of the present note.

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CITED LITERATURE

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[2] V. V. Nemytskii, V. V. Stepanov, *Qualitative Theory of Differential Equations*, Moscow–Leningrad, 1949.
[3] M. V. Bebutov, *Bull. Moscow State Univ.*, **2**, 5 (1941).

Note: Figure translations are in progress. See original paper for figures.

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