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Abstract

Full Text

MATHEMATICS

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ON THE PREDICATE OF DIFFERENTIABILITY AND THE OPERATOR OF DIFFERENTIATION IN CONSTRUCTIVE MATHEMATICAL ANALYSIS

(Presented by Academician P. S. Novikov on VI 25, 1962)

1. In this article we use the concepts and results of the theory of algorithms and of constructive mathematical analysis^(1,2). The following definitions will play the main role.

Let \mathfrak{A} be a sequence of rational numbers.* An algorithm \mathfrak{B} is called a **regulator of convergence in itself** of the sequence \mathfrak{A} if it transforms each natural number into a natural number and

$$\forall kmn (m, n \geq \mathfrak{B}(k) \supset |\mathfrak{A}(m) - \mathfrak{A}(n)| < 2^{-k}).$$

F-numbers are rational numbers and all possible words of the form P° , where P is the notation of a sequence of rational numbers for which one can construct a regulator of convergence in itself. **FR-numbers (duplexes)** are rational numbers and all possible words of the form $P^\circ Q$ such that P is the notation of a sequence of rational numbers, and Q is the notation of a regulator of convergence in itself of this sequence.

A **constructive metric space** is any three-term list (A, \mathfrak{P}, ρ) , where A is an alphabet, \mathfrak{P} is a one-place predicate specifying a set of words in the alphabet A , and ρ is an algorithm that is a metric function on the set \mathfrak{P} (the algorithm ρ transforms each pair of elements of the set \mathfrak{P} into a duplex). We shall denote the metric space (A, \mathfrak{P}, ρ) by \mathfrak{M} . Points X and Y of the space \mathfrak{M} are said to be **equal** if $\rho(X \square Y) = 0$.

Let \mathfrak{D} be a sequence of points of the space \mathfrak{M} , and let X be a point of the space \mathfrak{M} . An algorithm \mathfrak{C} is called a **regulator of convergence** of the sequence \mathfrak{D} to the point X if \mathfrak{C} transforms each natural number into a natural number and

$$\forall km (m \geq \mathfrak{C}(k) \supset \rho(\mathfrak{D}(m) \square X) < 2^{-k}).$$

The sequence \mathfrak{D} is called **convergent** to the point X if one can construct its regulator of convergence to X .

Theorem 1. *If in the space \mathfrak{M} there are at least two distinct points, then, whatever the point Z of this space may be, no algorithm is possible which transforms the notation of any sequence of points of the space \mathfrak{M} converging to Z into the notation of its regulator of convergence to the point Z .** *

* In constructive mathematics, a sequence of objects of a certain type is any algorithm that transforms each natural number into an object of this type.

** This theorem does not contradict the obvious assertion that, whatever the sequence of points of a metric space converging to a point Z may be, there exists a regulator of convergence of this sequence to the point Z . The point is that the constructive interpretation of this judgment asserts the existence of an algorithm constructing a regulator of convergence from a pair of initial data—the sequence itself and its regulator of convergence to the point Z .

Proof. Suppose that the condition of the theorem is satisfied, and let Z be a point of the space \mathfrak{M} . Using the principle of constructive choice (see (3)), one can construct a point U of the space \mathfrak{M} and a natural number i such that

$$\rho(U \square Z) > 2^{-i}.$$

Let \mathfrak{C} be an algorithm for which the problem of recognizing applicability to natural numbers is undecidable. We construct an algorithm ω having the following property: whatever natural numbers m and n may be,

$$\tilde{\omega}_{n \square}(m) \simeq \begin{cases} U, & \text{if the process of applying } \mathfrak{C} \text{ to } n \text{ ended in exactly } m \text{ steps}^*, \\ Z, & \text{otherwise.} \end{cases}$$

For any n , the algorithm $\tilde{\omega}_{n \square}$ represents a sequence of points of the space \mathfrak{M} . If \mathfrak{C} is applicable to n and the process of applying \mathfrak{C} to n terminates in q steps, then the algorithm that transforms any natural number into $q + 1$ serves as a regulator of convergence of the sequence $\tilde{\omega}_{n \square}$ to the point Z . If \mathfrak{C} is inapplicable to n , then an algorithm that transforms any natural number into 1 serves as a regulator of convergence of $\tilde{\omega}_{n \square}$ to Z . Using the provability in constructive logic of any formula of the type $((A \vee \neg A) \supset B) \supset B$, we obtain that, for every n , the sequence $\tilde{\omega}_{n \square}$ cannot fail to be convergent to Z .

Let \mathfrak{F} be an algorithm applicable to the record of any sequence convergent to the point Z , and constructing from it a record of a regulator of convergence of this sequence to the point Z . Note that the assertion “the algorithm \mathfrak{C} is a regulator of convergence of the sequence \mathfrak{D} to the point Z ” is equivalent to its double negation. Using this fact, and also the principle of constructive choice and the rules of constructive mathematical logic, it is easy to show that the algorithm

\mathfrak{F} is applicable to the record of any sequence that cannot fail to converge to the point Z , and constructs from it a record of a regulator of convergence of this sequence to the point Z . Construct an algorithm \mathfrak{G} such that

$$\{\mathfrak{G}_{n\Box}\} \simeq \mathfrak{F}(\{\tilde{\omega}_{n\Box}\})^{**}.$$

Let i be such that $\rho(Z\Box U) > 2^{-i}$. Then, for any m and n , from the inequality $m \geq \mathfrak{G}_{n\Box}(i)$ it follows that $\tilde{\omega}_{n\Box}(m) = Z$. Therefore the algorithm \mathfrak{C} is applicable to the number n if and only if there is an m not exceeding $\mathfrak{G}_{n\Box}(i)$ for which $\tilde{\omega}_{n\Box}(m) \neq Z$. The condition $\tilde{\omega}_{n\Box}m \neq Z$ is algorithmically checkable, since the duplex $\rho(\tilde{\omega}_{n\Box}(m)\Box Z)$ does not lie in the interval $(0, 2^{-i})$. From these considerations one can construct an algorithm deciding the problem of recognizing the applicability of the algorithm \mathfrak{C} to natural numbers. But the algorithm \mathfrak{C} was chosen so that such an algorithm is impossible. The theorem is proved.

Corollary 1. *If a metric space \mathfrak{M} contains at least two distinct points, then there is no algorithm constructing, from an arbitrary F -construct** in the space \mathfrak{M} , an FR -construct equal to it.*

Remark. Corollary 1 is a generalization of G. S. Tseitin's theorem⁽⁵⁾ on the connection between the notions of an F -number and an FR -number.

Corollary 2. *If \mathfrak{M} is a complete metric space containing at least two distinct points, then there is no algorithm constructing, from an arbitrary fundamental sequence of points of the space \mathfrak{M} , a point that is the limit of this sequence.*

* If \mathfrak{A} is an algorithm in the alphabet Σ and P is a word in the same alphabet, then by $\tilde{\mathfrak{A}}_P$ we shall denote, following⁽⁴⁾, such an algorithm that $\tilde{\mathfrak{A}}_P(Q) \simeq \mathfrak{A}(PQ)$ for any word Q in the alphabet Σ .

** $\{\mathfrak{A}\}$ denotes the record of the algorithm \mathfrak{A} .

*** The notions of an F -construct and an FR -construct in the space \mathfrak{M} are defined analogously to the notions of an F -number and an FR -number.

2. Let f_1 and f_2 be constructive functions transforming duplexes to which they are applicable into duplexes. In constructive analysis the predicates "the duplex y is a derivative number of the function f_1 at the point x " and " f_2 is a derivative of f_1 " are defined almost as in classical analysis, namely: one says that y is a derivative number of the function f_1 at the point x if f_1 is defined at the point x and for every natural number i one can construct a natural number j such that, for every point z whose distance from x is not greater than 2^{-j} and such that f_1 is defined at z , the inequality

$$|f_1(x) - f_1(z) - y(x-z)| \leq |x-z| \cdot 2^{-i}$$

is satisfied; one says that f_2 is a derivative of f_1 on the interval $[a, b]$ if, for every point x in $[a, b]$, the duplex $f_2(x)$ is a derivative number of the function f_1 at the point x . We shall say that f_1 is differentiable on $[a, b]$ if there exists a function f_2 that is a derivative of the function f_1 on $[a, b]$ (see (4)).

Theorem 2. *There is no algorithm which, from a record of an arbitrary function f differentiable on $[0, 1]$, constructs a record of some function that is a derivative of the function f on $[0, 1]$.*

The proof is based on Corollary 1 of Theorem 1 and on the following lemma.

Lemma. *Let \mathfrak{P}_0 be the set consisting of the two elements 0 and 1, and let ρ be an algorithm in the alphabet A , containing 0 and 1, such that $\rho(0\Box 0) = \rho(1\Box 1) = 0$, $\rho(1\Box 0) = \rho(0\Box 1) = 1$. One can construct an algorithm \mathfrak{f} such that, whatever the F -construct α in the space $(A, \mathfrak{P}_0, \rho)$, first, $\mathfrak{F}_{\alpha\Box}$ is a function differentiable on $[0, 1]$, and, second, if g is any function that is a derivative of $\mathfrak{F}_{\alpha\Box}$ on $[0, 1]$, then $g(0) = \alpha$.*

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- ¹ A. A. Markov, Tr. Matem. inst. im. V. A. Steklova AN SSSR, **42** (1954).
- ² N. A. Shanin, *ibid.*, **67** (1962), in press.
- ³ A. A. Markov, *ibid.*, **67** (1962), in press.
- ⁴ G. S. Tseitin, *ibid.*, **67** (1962), in press.

* However, one can construct an algorithm D which constructs a record of some function that is a derivative of a function f , from a pair of initial data—a record of the function f and a record of its differentiability regulator (the concept of differentiability regulator is introduced in a natural way). In constructive mathematical analysis the algorithm D is called the differentiation operator.

Note: Figure translations are in progress. See original paper for figures.

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