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# Mathematics

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## Abstract

## Full Text

Mathematics

M. Arato, Academician A. N. Kolmogorov, Ya. G. Sinai

# On the Estimation of Parameters of a Complex Stationary Gaussian Markov Process

§ 1. We shall consider a two-dimensional stationary random process whose components  $\xi(t)$  and  $\eta(t)$  satisfy the stochastic differential equations

$$\begin{aligned} d\xi &= -\lambda\xi dt - \omega\eta dt + d\varphi, \\ d\eta &= \omega\xi dt - \lambda\eta dt + d\psi, \end{aligned} \quad (1)$$

where  $\varphi(t)$  and  $\psi(t)$  are two independent Wiener processes with

$$M d\varphi = M d\psi = 0, \quad M(d\varphi)^2 = M(d\psi)^2 = a dt.$$

Putting

$$\zeta = \xi + i\eta, \quad \chi = \varphi + i\psi, \quad \gamma = \lambda - i\omega,$$

one can write system (1) in the form of a single equation

$$d\zeta = -\gamma\zeta dt + d\chi. \quad (1a)$$

The complex correlation function of our process has the form

$$C(\tau) = A(\tau) + iB(\tau) = M[\zeta(t)\overline{\zeta(t+\tau)}] = \sigma^2 \exp(-\lambda|\tau| - i\omega\tau), \quad (2)$$

where  $\sigma^2 = a/\lambda$ .

If the process is observed on the interval  $[0, T]$ , then one can determine the empirical correlation function

$$c(\tau) = a(\tau) + ib(\tau) = \frac{1}{T-\tau} \int_0^{T-\tau} \zeta(t)\overline{\zeta(t+\tau)} dt. \quad (3)$$

The empirical correlation function, with probability one, has at zero the right derivative

Fig. 1.  $\tau = 0.1 \cdot n; n = 0, 1, \dots, 156$

Figure 1: Fig. 1.  $\tau = 0.1 \cdot n; n = 0, 1, \dots, 156$

$$c'(0) = -a - \frac{1}{T}s_1^2 + \frac{1}{T}s_2^2 - ir,$$

where  $a$  is the parameter introduced above, characterizing the intensity of the “white noises”  $\varphi'(t)$  and  $\psi'(t)$ , and

$$s_1^2 = \frac{1}{2} [|\zeta(0)|^2 + |\zeta(T)|^2], \quad s_2^2 = \frac{1}{T} \int_0^T |\zeta(t)|^2 dt, \quad r = \frac{1}{T} \int |\zeta(t)|^2 d\theta.$$

The integration in the expression for  $r$  is carried out with respect to the angular argument  $\theta$ , determined from

$$\zeta(t) = |\zeta(t)|e^{i\theta(t)}.$$

In Fig. 1 the empirical correlation function  $c(\tau)$  is shown for Chandler variations of the coordinates of the Earth’ s pole\*.

\* The instantaneous axis of rotation of the Earth moves relative to the minor axis of the terrestrial ellipsoid (the so-called “free nutation”). In these displacements there is a periodic component with a one-year period. After their elimination there remain Chandler displacements, which have a tendency toward oscillations with a period of the order of 14 months, but are not strictly periodic, and with large, mainly smooth (waves of the order of 10-20 years), changes in amplitude. Fig. 1 shows that the Chandler component of the displacement of the pole satisfies well the hypotheses set out at the beginning of our note.

§ 2. The parameter  $a$  is determined exactly from the realization. It remains to consider the problem of estimating the parameters  $\lambda$  and  $\omega$ . Let  $P$  denote the probability measure in the space of realizations of our process on the interval  $[0, T]$ . In the same space introduce the standard measure

$$V = L \times W,$$

where  $L$  is the ordinary Lebesgue measure in the plane of  $\xi(0)$ , and  $W$  is the two-dimensional Wiener measure in the space of increments  $\xi(t) - \xi(0)$  with those characteristics

Fig. 1.  $\tau = 0.1 \cdot n; n = 0, 1, \dots, 156$

that were adopted for the random process  $x(t)$ . It can be shown that (cf. (2.3))

$$\frac{dP}{dV} = C\lambda \exp \left[ -\frac{\lambda^2 + \omega^2}{2a} T s_2^2 - \frac{\lambda}{a} s_1^2 + \lambda T + \frac{\omega}{a} T r \right], \quad (4)$$

where  $C$  is some constant. Formula (4) shows that the system of three statistics  $s_1^2, s_2^2, r$  is a sufficient system of statistics for the problem. Differentiating

$$L = \log \frac{dP}{dV} = c' + \log \lambda - \frac{\lambda^2 + \omega^2}{2a} T s_2^2 - \frac{\lambda}{a} s_1^2 + \lambda T + \frac{\omega}{a} T r$$

with respect to  $\omega$  and  $\lambda$ , we obtain the equations

$$\frac{\partial L}{\partial \omega} = -\frac{\omega}{a} T s_2^2 + \frac{T}{a} r = 0; \quad (5)$$

$$\frac{\partial L}{\partial \lambda} = \frac{1}{\lambda} - \frac{\lambda}{a} T s_2^2 - \frac{s_1^2}{a} + T = 0 \quad (6)$$

for determining the maximum-likelihood estimates  $\hat{\omega}$  and  $\hat{\lambda}$ . From (5) we obtain

$$\hat{\omega} = \frac{r}{s_2^2}. \quad (*)$$

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Fig. 1 was obtained by processing the data of Table 6 in the work of A. Ya. Orlov<sup>(1)</sup>. From the coordinates  $x(t), y(t)$  of Table 6, the component with an annual period was separated out, and the remainder was taken as  $\xi(t)$  and  $\eta(t)$ . In the figure, crosses indicate the points corresponding to increments  $\tau$  of 0.1 year. From the figure one can immediately see that the period  $2\pi/\omega$  is approximately equal to 14 months. The correct character of the spiral obtained may lead to the supposition that the parameter  $\lambda$  also admits a very accurate estimate. This, however, is probably not so, as will be explained at the end of the note. A more detailed account of the computational technique and discussion of the results will be published elsewhere.

It can be shown that

$$\frac{\hat{\omega} - \omega}{\sigma(\hat{\omega})}, \quad \sigma^2(\hat{\omega}) = \frac{a}{T s_2^2}$$

is distributed normally as  $(0, 1)$  (this is an exact, not an asymptotic result). Equation (6) always has a unique positive solution. Denoting  $\lambda T = \varkappa$ ,  $\hat{\lambda} T = \hat{\varkappa}$ , we obtain for  $\hat{\varkappa}$  the equation

$$h_2 \hat{\varkappa}^2 + (h_1 - 1) \hat{\varkappa} - 1 = 0,$$

where  $h_1 = s_1^2/aT$ ,  $h_2 = s_2^2/aT$ .

The distribution of the statistics  $h_1$  and  $h_2$ , and consequently of  $\hat{\nu}$ , depends only on the parameter  $\nu$ . Since the distribution of  $\hat{\nu}$  is continuous, for any  $\alpha$ ,  $0 < \alpha < 1$ , and  $\nu$ ,  $0 < \nu < \infty$ , one can find such a  $k$  that

$$\mathbf{P}\{\hat{\nu} > k \mid \nu\} = \alpha. \quad (7)$$

Inverting the dependence

$$k = k_\alpha(\nu),$$

we obtain the dependence

$$\nu = \nu_\alpha(k)$$

(it has been established that, as  $\nu$  varies from 0 to  $\infty$ , the function  $k_\alpha(\nu)$  varies monotonically from 0 to  $\infty$ , so that inversion is possible and unique). Obviously,

$$\mathbf{P}\{\nu < \nu_\alpha(\hat{\nu}) \mid \nu\} \equiv \alpha. \quad (8)$$

We have arranged the computation of the function  $\nu_\alpha(\hat{\nu})$  for  $\alpha = 0.1; 0.05; 0.025; 0.01; 0.005; 0.001; 0.9; 0.95; 0.975; 0.99; 0.995; 0.999$ . The results will be published after completion of the computations.

For small  $\hat{\nu}$ , (8) is equivalent to the relation

$$\mathbf{P}(\hat{\nu} < u\nu) = \exp\left(-\frac{1}{u}\right), \quad (9)$$

i.e. the ratio  $\hat{\nu}/\nu$  has a  $\chi^2$  distribution with two degrees of freedom. For large  $\hat{\nu}$ , (8) is equivalent to

$$\mathbf{P}(\nu < \hat{\nu} + u\sqrt{\hat{\nu}}) \simeq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^u e^{-t^2/2} dt, \quad (10)$$

i.e. the estimate  $\hat{\nu}$  is asymptotically normal with variance

$$\sigma^2(\hat{\nu}) \sim \hat{\nu}. \quad (11)$$

§ 3. For the case of the motion of the Earth's pole mentioned at the beginning of the note, on the basis of observations over  $T = 60$  years, the following were obtained\*

$$\hat{\omega} = 5.274, \quad \hat{\eta} = 3.6, \quad 2\pi : \hat{\omega} = 1.191, \quad \sigma(2\pi : \hat{\omega}) = 0.006.$$

\* Introducing Wiener  $\varphi$  and  $\psi$  into equations (1), i.e. perturbations of the “white-noise” type, is, of course, a rough idealization in the case of the motion of the Earth’ s poles. It would be more correct to write

$$\xi' = -\lambda\xi - \omega\eta + f, \quad \eta' = \omega\xi - \lambda\eta + g.$$

However, the data from (1) show that the values of the functions  $f(t)$  and  $g(t)$  at time instants  $t$  separated from one another by several years are practically mutually independent, so that replacing the functions  $f$  and  $g$  by “equivalent white noise” is legitimate. The error in determining the intensity of this equivalent white noise is, apparently, sufficiently small not to affect substantially the results of estimating the parameter  $\lambda$ . The value of  $\hat{\omega}$  was calculated by the discrete analogue of formula (\*), obtained by the method of maximum likelihood for a “scheme with discrete time.”

For estimation of the parameters  $\lambda$  and  $\omega$  for the motion of the Earth’ s pole, see also (6). In (6), results close to ours are given:  $\lambda = 1/15$ ,  $2\pi : \omega = 1.193$ . Close values were indicated by Jeffreys (7), but in papers (5, 8) sharply different values  $\lambda = 0.3$  and  $\lambda = 0.01$  were indicated. The reasons for these discrepancies will be explained elsewhere.

The asymptotic formula (11) gives

$$\sigma^2(\hat{\chi}) = 3.6.$$

Since  $\chi$  is known to be positive, while formula (10), for  $\alpha < 0.03$ , yields a negative estimate of  $k_\alpha$ , it is clear that the asymptotic formula (10) is not yet applicable.

The calculation carried out by us yields the estimates

$$\begin{array}{lll} \chi_{0.90} = 5.5, & \chi_{0.95} = 6.2, & \chi_{0.975} = 7.8, \\ \chi_{0.10} = 1.27, & \chi_{0.05} = 0.82, & \chi_{0.025} = 0.46, \end{array}$$

which correspond, for  $\lambda$ , to the estimates

$$\begin{array}{lll} \lambda_{0.90} = 0.09, & \lambda_{0.95} = 0.10, & \lambda_{0.975} = 0.13, \\ \lambda_{0.10} = 0.02, & \lambda_{0.05} = 0.01, & \lambda_{0.025} = 0.008. \end{array}$$

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*Note: Figure translations are in progress. See original paper for figures.*

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