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# MATHEMATICAL PHYSICS

1962

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**Abstract**

**Full Text**

## MATHEMATICAL PHYSICS

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### REDUCTION OF SOLUTIONS OF DIFFERENTIAL EQUATIONS OF ELLIPTIC TYPE

*(Presented by Academician N. N. Bogolyubov on 16 I 1962)*

Let  $u$  be a regular solution of the elliptic differential equation (2). For  $c \leq 0$ ,  $u$  either is a constant or attains its maximum value only on the boundary of the domain. Therefore, if  $u$  is not equal to a constant, it cannot be regular at all points of the  $m$ -dimensional Euclidean space  $E_m$ ; there exist domains  $G_j$ ,  $j = 1, 2, \dots, n$ , in which  $u$  is not defined. The surface bounding  $G_j$  will be denoted by  $S_j$ . Suppose that  $S_j$  is a piecewise smooth surface. We choose the closed domains  $G_j + S_j$  so that they have no common points. Denote by  $G_0$  the domain containing in its interior all the closed domains  $G_j + S_j$ ,  $j = 1, 2, \dots, n$ . The surface  $S_0$  bounding  $G_0$  will be assumed piecewise smooth. In the domain  $G_0 + S_0 - \sum_{j=1}^n G_j$ , the solution  $u$  must be regular. Represent  $u$  as a sum of terms

$$u = u_0 + \sum_{j=1}^n u_j, \quad (1)$$

where  $u_0$  is a solution of (2) regular in  $G_0$ , and each of the  $u_j$ ,  $j = 1, 2, \dots, n$ , is a solution of (2) that is regular in  $G_0 - (G_j + S_j)$  and has a singularity at least at one point of the domain  $G_j$ .

The domains  $G_j$ ,  $j = 1, 2, \dots, n$ , will be called nests; the functions  $u_0, u_1, u_2, \dots, u_n$  components, or reduction components\*, and the representation of  $u$  as a sum of reduction components—a reduction. Each nest can in general be divided into several nests. If the reduction is such that any further increase in the number of nests is impossible, it is called complete, and its components are called elementary.

In this paper we shall study the question of reducing solutions of differential equations of elliptic type and shall formulate Theorems 1-5, which are essentially generalizations of the results of the work (1).

**Theorem 1.** *Let the first derivatives of the functions  $a_{ik}$ ,  $e_i$ ,  $i = 1, 2, \dots, n$ , be continuous in  $G_0 + S_0$ , and let Green's functions exist for equations (2) and (3) in the domain  $G_0 + S_0$ . Then every nonconstant solution of (2), regular in  $G_0 + S_0 - \sum_{j=1}^n G_j$ , is reducible.*

**Proof.** Denote by  $M$  the elliptic operator

$$M = \sum_{i,k=1}^m \frac{\partial}{\partial x_k} \left( a_{ik} \frac{\partial}{\partial x_i} \right) + \sum_{i=1}^m e_i \frac{\partial}{\partial x_i} + c$$

and consider the function  $u = u(x_1, x_2, \dots, x_m)$ , which in the domain  $G_0 - \sum_{j=1}^n G_j$  is a regular solution of the equation

$$Mu = 0. \quad (2)$$

\* If several components  $u'_j, u''_j, \dots$  belong to one nest  $G_j$ , in (1) only their sum  $u_j$  is taken into account.

We shall also need the equation ( $N$  is the operator adjoint to  $M$ )

$$Nv = 0. \quad (3)$$

Denote by  $u^*(P, Q)$  the Green's function of equation (3) and of the domain  $G_0 + S_0$ . We shall denote the variable point by  $Q$ , and the fixed one by  $P$ . We agree that the differential equation is satisfied with respect to the variable that is written in the second position. For example,  $u^*(P, Q)$  satisfies (2) with respect to the variable  $Q$ . We denote the analogous Green's function for (2) by  $v^*(P, Q)$ . The functions  $u^*$  and  $v^*$  are normalized by the condition

$$\int_{(G_0)} Mu^*(P, Q) d\tau = \int_{(G_0)} Nv^*(P, Q) d\tau = -1,$$

where  $d\tau$  is the differential element of the  $m$ -dimensional Euclidean space at the point  $Q$ . Apply Green's formula in the closed domain  $G_0 + S_0 - \sum_{j=1}^m G_j$ :

$$\begin{aligned} & \int_{(G_0+S_0-\sum G_j)} \{v^*(P, Q)Mu - uN[v^*(P, Q)]\} d\tau = \\ & = \int_{(S_0+\sum_{j=1}^n S_j)} \left\{ a \left[ v^*(P, Q) \frac{\partial u}{\partial \nu} - u \frac{\partial v^*(P, Q)}{\partial \nu} \right] + b u v^*(P, Q) \right\} d\sigma. \end{aligned}$$

Here

$$a = \left[ \sum_{i=1}^m \left( \sum_{k=1}^m a_{ki} \gamma_k \right)^2 \right]^{1/2}; \quad b = \sum_{i=1}^m e_i \gamma_i;$$

$\gamma_i$  are the direction cosines of the exterior normal to the surfaces  $S_0$  and  $S_j$ ,  $j = 1, 2, \dots, n$ ;  $d\sigma$  is the surface element;  $\nu$  is the conormal.

For  $P \in G_0 - \sum_{j=1}^n G_j$ , Green's formula leads to the relation

$$u(P) = u_0(P) + \sum_{j=1}^n u_j(P),$$

where

$$\begin{aligned} u_0(P) &= \int_{(S_0)} \left\{ a \left[ v^*(P, Q) \frac{\partial u}{\partial \nu} - u \frac{\partial v^*(P, Q)}{\partial \nu} \right] + buv^*(P, Q) \right\} d\sigma, \\ u_j(P) &= - \int_{(S_j)} \left\{ a \left[ v^*(P, Q) \frac{\partial u}{\partial \nu} - u \frac{\partial v^*(P, Q)}{\partial \nu} \right] + buv^*(P, Q) \right\} d\sigma, \end{aligned} \quad (4)$$

which is identical with the reduction formula (1). It remains only to prove that the terms  $u_0, u_1, u_2, \dots, u_n$  are reduction components. Indeed, from the adjointness of  $M$  and  $N$  it follows that  $v^*(P, Q) = u^*(Q, P)$ . Therefore

$$\begin{aligned} u_0(P) &= \int_{(S_0)} \left\{ a \left[ u^*(Q, P) \frac{\partial u}{\partial \nu} - u \frac{\partial u^*(Q, P)}{\partial \nu} \right] + buu^*(Q, P) \right\} d\sigma, \\ u_j(P) &= - \int_{(S_j)} \left\{ a \left[ u^*(Q, P) \frac{\partial u}{\partial \nu} - u \frac{\partial u^*(Q, P)}{\partial \nu} \right] + buu^*(Q, P) \right\} d\sigma. \end{aligned} \quad (5)$$

The Green's function  $u^*(Q, P)$  satisfies (2) with respect to  $P$ . From (5) it follows that  $u_0(P)$  is a regular solution of (2) in  $G_0 - (G_j + S_j)$ , while  $u_j(P)$  is a regular solution of (2) in  $G_0 - (G_j + S_j)$ , i.e.  $u_0, u_1, \dots, u_n$  are reduction components. The theorem is thereby proved.

**Theorem 2.** Let the domains  $G'_j, G''_j$ , bounded by the surfaces  $S'_j, S''_j$ , contain within them  $G_j + S_j$ , and let the closed domains  $G'_j + S'_j, G''_j + S''_j$  have no common points with the remaining nests  $G_k + S_k, k = 1, 2, \dots, j-1, j+1, \dots, n$ . Let  $u'_j, u''_j$  be the values obtained from formula (4), in which  $S'_j, S''_j$ , respectively, have been substituted for  $S_j$ . Then  $u'_j = u''_j$  for all points lying outside  $S'_j$  and outside  $S''_j$ .

**Theorem 3.** Let the surface  $S_k$  bound the domain  $G_k$ , in which  $u$  is regular. Then  $u_k = 0$ .

**Theorem 4.** Suppose that, for a given system of nests  $G_0, G_j, j = 1, 2, \dots, n, u$  is reduced both by means of the functions  $u_0, u_j, j = 1, 2, \dots, n$ , and by means of the functions  $\tilde{u}_0, \tilde{u}_j, j = 1, 2, \dots, n$ . Then, if  $c \leq 0$ , there exist solutions (2),  $w_j, j = 1, 2, \dots, n$ , which are regular in  $G_0$  and in  $G_0 - (G_j + S_j)$ , and coincide with  $u_j - \tilde{u}_j$ .

**Proof.** Substituting (1) into (4), we find

$$u_j(P) = - \int_{(S_j)} \left\{ a \left[ v^*(P, Q) \frac{\partial u_j}{\partial \nu} - u_j \frac{\partial v^*(P, Q)}{\partial \nu} \right] + b u_j v^*(P, Q) \right\} d\sigma,$$

$$P \in G_0 - \sum_{j=1}^n (G_j + S_j).$$

In deriving this formula it is taken into account that, for  $i \neq j$ , the solution  $u_i$  is regular in  $G_j + S_j$ . Therefore it follows from Theorem 3 that all integrals of the form

$$\int_{(S_j)} \left\{ a \left[ v^*(P, Q) \frac{\partial u_i}{\partial \nu} - u_i \frac{\partial v^*(P, Q)}{\partial \nu} \right] + b u_i v^*(P, Q) \right\} d\sigma, \quad i \neq j,$$

are equal to zero. Substituting into (4) the function

$$u = \tilde{u}_0 + \sum_{j=1}^n \tilde{u}_j,$$

we similarly find that

$$u_j(P) = - \int_{(S_j)} \left\{ a \left[ v^*(P, Q) \frac{\partial \tilde{u}_j}{\partial \nu} - \tilde{u}_j \frac{\partial v^*(P, Q)}{\partial \nu} \right] + b \tilde{u}_j v^*(P, Q) \right\} d\sigma,$$

$$P \in G_0 - \sum_{j=1}^n (G_j + S_j),$$

$u_j(P)$  being obtained once by means of a surface integral involving  $u_j$ , and a second time by means of a surface integral involving  $\tilde{u}_j$ . Subtracting the corresponding expressions, we obtain

$$\int_{(S_j)} \left\{ a \left[ v^*(P, Q) \frac{\partial}{\partial \nu} (u_j - \tilde{u}_j) - (u_j - \tilde{u}_j) \frac{\partial v^*(P, Q)}{\partial \nu} \right] \right.$$

$$\left. + b (u_j - \tilde{u}_j) v^*(P, Q) \right\} d\sigma = 0, \quad P \in G_0 - \sum_{j=1}^n (G_j + S_j). \quad (6)$$

The function

$$w_j(P) = \int_{(S_0)} \left\{ a \left[ u^*(Q, P) \frac{\partial}{\partial \nu} (u_j - \tilde{u}_j) - (u_j - \tilde{u}_j) \frac{\partial u^*(Q, P)}{\partial \nu} \right] + bu^*(Q, P)(u_j - \tilde{u}_j) \right\} d\sigma, \quad P \in G_0, \quad (7)$$

is a regular solution of (2) in  $G_0$ . We apply Green's formula

in the region  $G_0 - \sum_{j=1}^n (G_j + S_j)$  for the functions  $u_j - \tilde{u}_j$  and  $v^*(P, Q)$ :

$$u_j - \tilde{u}_j = \int_{(S_0)} \left\{ a \left[ v^*(P, Q) \frac{\partial}{\partial \nu} (u_j - \tilde{u}_j) - (u_j - \tilde{u}_j) \frac{\partial v^*(P, Q)}{\partial \nu} \right] + b(u_j - u_j)v^*(P, Q) \right\} d\sigma - \int_{(S_j)} \left\{ a \left[ v^*(P, Q) \frac{\partial}{\partial \nu} (u_j - \tilde{u}_j) - (u_j - \tilde{u}_j) \frac{\partial v^*(P, Q)}{\partial \nu} \right] + b(u_j - u_j)v^*(P, Q) \right\} d\sigma. \quad (8)$$

Substituting (6) and (7) into (8) (before substituting (7) we replace  $u^*(Q, P)$  by  $v^*(P, Q)$ ), we find

$$\omega_j(P) = u_j(P) - \tilde{u}_j(P), \quad P \in G_0 - \sum_{j=1}^n (G_j + S_j).$$

But since  $c \leq 0$ , the equality is also valid in the broader region

$$\omega_j(P) = u_j(P) - \tilde{u}_j(P), \quad P \in G_0 - (G_j + S_j).$$

The obtained equality and (7) prove the theorem.

In certain cases the boundary conditions are determined uniquely by the nature of the problem. For example, the bioelectric potential  $U$  satisfies (2), with  $a_{ik} = \delta_{ik}\lambda$ , where  $\delta_{ik}$  is the Kronecker symbol, and the scalar function  $\lambda$  is the specific electrical conductivity of the tissues. If the exchange of electric charges with the surrounding medium may be neglected, then it is natural to put  $\partial u_j / \partial n = 0$  on  $S_0$ . This means that the reduction is unique. In particular, foci of electrical activity in the brain can be isolated uniquely, despite their mutual influence. Sometimes  $U$  may be determined in all space. We formulate a theorem for this case.

**Theorem 5.** *Let  $S_0$  be a ball of radius  $R_0$ , and suppose that as  $R_0 \rightarrow \infty$  the first derivatives  $a_{ik}$  and  $e_i$  are continuous in  $G_0 + S_0$ , and  $c \leq 0$ . If as  $R_0 \rightarrow \infty$  the Green's function exists and if  $u$ ,  $d$ , and also the reduced components tend to zero, then  $u$  is reduced uniquely.*

The theorems and formulas of the present paper may be useful in separating out the gravitational field of the body under consideration, if it is given together with the gravitational fields of other bodies, i.e. in solving the inverse problem of potential theory (<sup>1-4</sup>). They may be used in separating geophysical anomalies (<sup>1</sup>) and in the study of bioelectric phenomena.

The author expresses deep gratitude to I. G. Petrovskii for a number of valuable remarks made during discussion of this work.

Received  
12 I 1962

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*Note: Figure translations are in progress. See original paper for figures.*

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